1 Introduction:

In this chapter, we will analyze motion using the concepts of work and energy.

2 The work of a force:

In mechanics a force $\vec{F}$ does work on a particle only when the particle undergoes a displacement in the direction of the force.

- The work $dU$ done by a force $\vec{F}$ in displacing a particle $d\vec{r}$ is a scalar quantity defined by:

$$dU = \vec{F} \cdot d\vec{r} = F ds \cos(\theta) = F_{t} ds$$

where $ds = |d\vec{r}|$, $\theta$ is the angle between the tails of $d\vec{r}$ and $\vec{F}$, and $F_{t}$ is the component of the force $\vec{F}$ parallel to the displacement. Note that if $0 < \theta < 90^\circ$ the force component and the displacement have the same sense so the work is positive; whereas if $90^\circ < \theta < 180^\circ$, these vectors have an opposite sense, and therefore the work is negative.

- The unit of work is the unit of force times the unit of distance, specifically 1 J = 1 N · m

- **Work done by a variable force.** If a particle undergoes a finite displacement along its path from $\vec{r}_1$ to $\vec{r}_2$ or $s_1$ to $s_2$, the work done is given by

$$U_{1-2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{s_1}^{s_2} F \cos(\theta) ds$$

The integral in this equation can be interpreted as the area under the curve from position $s_1$ to position $s_2$. 
- **Work of a constant force moving along a straight line.** Since both \( \vec{F} \) and \( \theta \) are constant (straight line path)

\[
U_{1-2} = F \cos(\theta) \int_{s_1}^{s_2} ds = F \cos(\theta)(s_2 - s_1)
\]

- **Work of a weight.**

\[
U_{1-2} = -m \ g \Delta y = -m \ g (y_2 - y_1)
\]

where the vertical displacement \( \Delta y \) is measured positive upward (so the work of he weight is positive if the particle is displaced downward and negative if displaced upward).

- **Work of a spring force.**

  - Work done on the spring. Work done on the spring by the force \( \vec{F}_s \) where \( \vec{F}_s \) is in the same direction as the displacement of the free end of the spring and \( \vec{F}_s \cdot d\vec{r} = k \ s \ ds \), \( k \) is the spring constant of the spring.

\[
U_{1-2} = \int_{s_1}^{s_2} k \ s \ ds = \frac{1}{2} k (s_2^2 - s_1^2)
\]
Work done on a body (or particle) attached to a spring. In this case the force $\vec{F}_s$ exerted on the particle is opposite to that exerted on the spring. Hence, the force $\vec{F}_s$ will do negative work on the body (particle).

\[
U_{1-2} = \int_{s_1}^{s_2} -k \ s \ ds = -\frac{1}{2} \ k \ (s_2^2 - s_1^2)
\]

3 The principle of work and energy:

The resultant of all the forces acting on a particle (at any time) as it moves along its path is

\[
\vec{F}_R = \sum \vec{F} = (\sum F_i)\hat{u}_t + (\sum F_n)\hat{u}_n
\]
Recall that

\[ \sum F_t = m \ a_t = m \frac{dv}{dt} = m \frac{dv}{ds} \]

\[ \sum F_n = m \ a_n \]

The work done by all the forces acting on the particle as it moves from 1 to 2 is

\[ \sum U_{1-2} = \sum \int_{s_1}^{s_2} F_t \ ds = \int_{s_1}^{s_2} m v \frac{dv}{ds} \ ds = \int_{v_1}^{v_2} m v \ dv = \frac{1}{2} m (v_2^2 - v_1^2) \]

- The work and energy principle can be used to solve a variety of kinetic problems which involve force, velocity and displacement for example:

  - If a particle’s initial speed is known and the work of all the forces on the particle can be determined then the work energy principle provides a direct means of obtaining the final speed of the particle after it undergoes a specified displacement.

\[ v_2 = \sqrt{v_1^2 + \frac{2}{m} \sum U_{1-2}} \]

- The process would have been done in two steps if one opted to use the equations of motion; i.e., apply \( \sum F_t = m a_t \) to obtain \( a_t \), then integrate \( a_t = \frac{dv}{ds} \) to obtain \( v_2 \).

- The principle of work and energy cannot be used, in general, to determine forces directed normal to the path, since these forces do no work on the particle. Instead \( \sum F_n = m \ a_n \) must be used. For curved paths, however, the magnitude of the normal force is a function of speed. Hence it may be necessary to obtain this speed using the principle of work and energy, and then substitute this quantity into the equation of motion \( \sum F_n = m \frac{v^2}{\rho} \) to obtain the normal force.

3.1 Procedure of analysis:

For application of the work energy principle in solving kinetic problems it is suggested that the following procedure be used:

- **Work (Free-body diagram)**
  - Establish the inertial coordinate system and draw a free body diagram of the particle in order to account for all the forces that do work on the particle as it moves along its path.

- **Principle of Work and energy**
  - A force does work when it moves through a displacement in the direction of the force.
  - Work is positive when the force component is in the same direction as the displacement, otherwise it is negative.
  - Forces that are function of displacement must be integrated to obtain the work. Graphically, the work is equal to the area under the force-displacement curve.
  - The work of a weight is the product of the weight magnitude and the vertical displacement, \( U_w = \pm Wy \). it is positive when the weight moves downwards.
  - The work of a spring is of the form \( U_s = \frac{1}{2}ks^2 \), where \( k \) is the spring stiffness and \( s \) is the stretch or compression of the spring.
4 Principle of work and energy for a system of particles:

Consider a system of particles isolated within within an enclosed region of space as shown in the figure below.

Each particle $i$ is subjected to both external forces $\vec{F}_i$ and internal forces $\vec{f}_i$. The principle of work and energy applied to the $i^{th}$ particle is thus

$$\frac{1}{2} m_i v_{i1}^2 + \int_{s_{i1}}^{s_{i2}} (F_i)_t \, ds + \int_{s_{i1}}^{s_{i2}} (f_i)_t \, ds = \frac{1}{2} m_i v_{i2}^2$$

Summing over all the particles in the system gives

$$\sum \frac{1}{2} m_i v_{i1}^2 + \sum \int_{s_{i1}}^{s_{i2}} (F_i)_t \, ds + \sum \int_{s_{i1}}^{s_{i2}} (f_i)_t \, ds = \sum \frac{1}{2} m_i v_{i2}^2$$

or symbolically as

$$\sum T_1 + \sum U_{1-2} = \sum T_2$$

This equation states that the system’s initial kinetic energy ($\sum T_1$) plus the work done by the external and internal forces acting on the particles of the system ($\sum U_{1-2}$) is equal to the system final kinetic energy ($\sum T_2$).

- Note that although the internal forces on adjacent particles occur in equal but opposite collinear pairs, the total work done by each of these forces will, in general, not cancel out since the path over which corresponding particles travel will be different. There are two important exceptions to this rule which often occur in practice:
  
  - When particles are contained within the boundary of a translating rigid body
  - When particles are connected by inextensible cables

In these cases adjacent particles exert equal but opposite internal forces that have component which undergo the same displacement, and therefore the work of these forces cancels.
5  Power and efficiency:

• Power
  – Power \( P \) is defined as the amount of work performed per unit of time.
  – If a machine or engine performs a certain amount of work, \( dU \), within a given time interval, \( dt \), the power generated can be calculated as
    \[
    P = \frac{dU}{dt}
    \]
  – Since the work can be expressed as \( dU = \vec{F} \cdot d\vec{r} \), the power can be written
    \[
    P = \frac{dU}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \frac{\vec{F} \cdot \vec{V}}{dt} = FV \cos(\theta)
    \]
    where \( \theta \) is the angle between the force and velocity vectors
  – Thus, power is a scalar defined as the product of the force and velocity components acting in the same direction.
  – The unit of power in the SI system is the \textit{watt} (W) where
    \[
    1 \text{ W} = 1 \text{ J/s} = 1 \left( \frac{N \cdot m}{s} \right)
    \]
  – In the US customary system of units, power is usually expressed in units of horsepower (hp) where
    \[
    1 \text{ hp} = 550 \left( \frac{ft \cdot lb}{s} \right) = 746 \text{ W}
    \]

• Efficiency
  – The mechanical efficiency of a machine is the ratio of the useful power produced (output power) to the power supplied to the machine (input power) or
    \[
    \epsilon = \frac{\text{power output}}{\text{power input}}
    \]
  – If energy input and removal occur at the same time, efficiency may also be expressed in terms of the ratio of output energy to input energy or
    \[
    \epsilon = \frac{\text{energy output}}{\text{energy input}}
    \]
  – Machines will always have frictional forces. Since frictional forces dissipate energy, additional power will be required to overcome these forces. Consequently, the efficiency of a machine is always less than 1.

5.1 Procedure of analysis:

• Find the resultant external force acting on the body causing its motion. It may be necessary to draw a free-body diagram.
• Determine the velocity of the point on the body at which the force is applied. Energy methods or the equation of motion and appropriate kinematic relations, may be necessary.
• Multiply the force magnitude by the component of velocity acting in the direction of \( \vec{F} \) to determine the power supplied to the body (\( P = Fv \cos(\theta) \)).
• In some cases, power may be found by calculating the work done per unit of time (\( P = dU/dt \)).
• If the mechanical efficiency of a machine is known, either the power input or output can be determined.
6 Conservative forces and potential energy:

- **Conservative forces**
  
  - A force \( \vec{F} \) is said to be conservative if the work done is independent of the path followed by the force acting on a particle as it moves from A to B. In other words, the work done by the force \( \vec{F} \) in a closed path (i.e., from A to B and then back to A) equals zero.

  \[
  \oint \vec{F} \cdot d\vec{r} = 0
  \]

  This means the work is conserved.

- A conservative force depends only on the position of the particle, and is independent of its velocity or acceleration.
- The weight, the spring force acting on a particle are examples of conservative forces.
- In contrast the work done by the frictional forces depends on the path and consequently, frictional forces are non-conservative. The work is dissipated from the body in the form of heat.

- **Potential energy**
  
  - Energy can be defined as the ability to do work. When the energy comes from the motion of the particle, it is referred to as kinetic energy. When it comes from the position of the particle measured from a fixed reference or datum, it is called potential energy.
  
  - The potential energy is measured as the amount of work a conservative force will do when it moves from a given position to a reference datum.
  
  - **Gravitational potential energy**: \( V_g \)

  \[
  V_g = W \ y
  \]

  where \( W \) is the weight of the particle and \( y \) is the position of the particle measured positive upward from an arbitrary selected datum.
- Elastic potential energy: \( V_e \)

\[
V_e = \frac{1}{2} ks^2
\]

where \( k \) is the stiffness of the springs and \( s \) is the displacement from the unstretched position. Note that \( V_e \) is always positive since, in the deformed position, the force of the spring has the capacity for always doing positive work on the particle when the spring is returned to its upstretched position.

- In general, if a particle is subjected to both gravitational and elastic forces, the particle’s potential energy can be expressed as a potential function, which is the algebraic sum

\[
V = V_g + V_e
\]

- Proving a force \( \vec{F} \) is conservative

Consider a conservative force \( \vec{F} \) producing a potential energy \( V(x, y, z) \). The work done by this force in taking the particle over an infinitesimal displacement \( d\vec{r} \) is

\[
dU = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz
\]

on the other hand

\[
dU = V(x, y, z) - V(x + dx, y + dy, z + dz) = -dV(x, y, z)
\]
then

$$F_x dx + F_y dy + F_z dz = -dV(x, y, z) = -\left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right)$$

since changes in \(x, y, z\) are all independent of one another, the previous result is satisfied provided that

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

and

$$\vec{F} = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)V = -\nabla V$$

In other words if a force \(\vec{F}\) is conservative then there exist a potential energy function \(V\) such that \(\vec{F} = -\nabla V\)

7 Conservation of energy

When a particle is acted upon by a system of conservative and non-conservative forces, the portion of the work done by the conservative forces can be written in terms of the difference in their potential energies \((\sum U_{1-2})_{cons} = V_1 - V_2\) as a result the principle of work and energy can be written as:

$$T_1 + V_1 + (\sum U_{1-2})_{noncons} = T_2 + V_2$$

here \((\sum U_{1-2})_{noncons}\) represents the work of the non-conservative forces acting on the particle \(T_1, T_2\) are the initial and final kinetic energies. \(V_1, V_2\) are the initial and final potential energies. If only conservative forces are applied to the particle this term is zero and then we have

$$T_1 + V_1 = T_2 + V_2$$

This equation is referred to as the conservation of mechanical energy. It states that during the motion the sum of particle’s kinetic and potential energies remains constant. for this to occur, kinetic energy must be transformed into potential energy and vise versa.

- **System of particles.** If a system of particles is subjected to only conservative forces The equation of conservation of energy for the system becomes

$$\sum T_1 + \sum V_1 = \sum T_2 + \sum V_2$$

Here the sum of the system’s initial kinetic and potential energies is equal to the system’s final kinetic and potential energies. In other words, \(\sum T + \sum V = const\)
7.1 Solving problems using the Conservation of energy principle

The conservation of energy equation is used to solve problems involving velocity, displacement and conservative force systems. It is generally easier to apply than the principle of work and energy because the energy equation just requires specifying the particle’s kinetic and potential energies at only two points along the path, rather than determining the work when the particle moves through a displacement. For applications it is suggested that the following procedure be used.

- **Potential energy**
  - Draw two diagrams showing the particle located at its initial and final points along the path
  - If the particle is subjected to a vertical displacement, establish the fixed horizontal datum from which to measure the particle’s gravitational potential energy \( V_g \)
  - Data pertaining to the elevation \( y \) of the particle from the datum and the extension or compression \( s \) of any connecting springs can be determined from the geometry associated with the two diagrams
  - Recall \( V_g = W_y \), where \( y \) is positive upward from the datum and negative downward from the datum; also for a spring, \( V_c = \frac{1}{2} ks^2 \), which is always positive

- **Conservation of energy**
  - Apply the equation \( T_1 + V_1 = T_2 + V_2 \)
  - When determining the kinetic energy \( T = \frac{1}{2} mv^2 \), the particle’s speed must be measured from an inertial frame of reference.