I. Moment of Inertia:

Since a body has a definite size and shape, an applied nonconcurrent force system may cause the body to both translate and rotate. The translational aspects of the motion were studied in chapter 3 and are governed by the equation $\vec{F} = m \vec{a}$. It will be shown in this chapter that the rotational aspects, caused by a moment $\vec{M}$, are governed by an equation of the form $\vec{M} = I \vec{\alpha}$. The symbol $I$ in the equation is termed the moment of inertia. By comparison, the moment of inertia is a measure of the resistance of a body to angular acceleration ($\vec{M} = I \vec{\alpha}$) in the same way that mass is a measure of the body’s resistance to acceleration ($\vec{F} = m \vec{a}$).

We define the moment of inertia as an integral of the "second moment" about an axis of all the elements of mass $dm$ which compose the body. For example, the body’s moment of inertia about the z-axis in the figure below is

$$I = \int r^2 \, dm$$

Here the "moment arm" $r$ is the perpendicular distance from the z-axis to an element of mass $dm$. Clearly $I$ is always positive and has units of $kg \cdot m^2$ or $slug \cdot ft^2$.

![Moment of Inertia Diagram]

- If the body consists of material having a variable density, $\rho = \rho(x, y, z)$, we can write $dm = \rho \, dV$ and express $I$ in terms of volume:

$$I = \iiint r^2 \, \rho \, dV$$

If $\rho$ is constant (homogeneous solid), we can write

$$I = \rho \iiint r^2 \, dV$$

- When the elemental volume chosen for integration has infinitesimal dimensions in all three directions $dV = dx \, dy \, dz$, the moment of inertia of the body must be determined using "triple integration". The integration process can, however, be simplified to a single integration provided the chosen elemental volume has a differential size in only one direction. Shell or disk elements are often used for this purpose.
**Procedure for analysis:** To obtain the moment of inertia by integration, we will consider only symmetric bodies having surfaces which are generated by revolving a curve about an axis. An example of such a body which is generated about the z axis is shown in the figure below. Two types of differential elements can be chosen.

**– Shell element**

* If a shell element having a height \( z \), radius \( r = y \), and thickness \( dy \) is chosen for integration (see figure below), then the volume is \( dV = (2\pi y) z \, dy \)

**– Disk element**

* If a disk element having a radius \( y \) and a thickness \( dz \) is chosen for integration, then the volume is \( dV = (\pi y^2) \, dz \)

* This element is finite in the radial direction, and consequently its parts do not all lie at the same radial distance \( r = y \) from the z-axis. As a results the previous equation cannot be used to determine \( I_z \) directly. Instead, to perform the integration, it is first necessary to determine the moment of inertia of the element about the z-axis and then integrate this result.
• **Parallel axis theorem:** Consider the body in the figure below. The axis $z'$ pases through the mass center of the body. Consider an element of mass $dm$ in the body. we have the following relations $r^2 = (d+x')^2 + y'^2$ and $\int x'\,dm = \int y'\,dm = 0$.

![Parallel axis theorem diagram]

\[
I = \int_m r^2 \, dm = \int_m [(d + x')^2 + y'^2] \, dm
\]

\[
I = \int_m (x'^2 + y'^2) \, dm + 2d \int_m x' \, dm + d^2 \int_m dm
\]

\[
I = I_G + m\,d^2
\]

where $I_G = \int_m (x'^2 + y'^2) \, dm$ is the moment of inertial about the z’axis passing through the center of mass $G$, $m$ the mass of the body, and $d$ is the perpendicular distance between the parallel axes.

• **Radius of Gyration:** occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the radius of gyration $k$ such that

\[
I = m\,k^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}}
\]

• **Composite bodies:** If a body is constructed of a number of simple shapes such as disks, spheres and rods, the moment of inertia of the body about any axis $z$ can be found by adding algebraically the moment of inertia of all the composite shapes computed about the $z$-axis.

II. **Planar kinetic equations of motion:**

In the following analysis we will limit our study to planar kinetics to rigid bodies which, along with their loadings, are considered to be symmetrical with respect to a fixed reference plane. In this case the path of motion of each particle of the body is a plane curve parallel to a fixed reference plane. Since the motion of the body may be viewed within the reference plane, all the forces (and couple moments) acting on the body can then be projected onto the plane. An example of an arbitrary body of this type is shown in the figure below. Here the inertial frame of reference $x, y, z$ has its origin coincident with the arbitrary point $P$ in the body. By definition these axes do not rotate and are either fixed or translate with constant velocity.

![Planar kinetic equations of motion diagram]
II-1. Equations of translational motion:

In chapter 3 we have shown that for a system of particles Newton’s second law can be written as

\[ \sum \vec{F} = m \vec{a}_G \]

This equation is refereed to as the translational equation of motion for the mass center of a rigid body. It states that the sum of all the external forces acting on the body is equal to the body’s mass times the acceleration of its center of mass G.

For motion on the body in the x-y plane, the translational equation of motion may be written in the form of two independent scalar equations namely,

\[ \sum F_x = m (a_G)_x \]
\[ \sum F_y = m (a_G)_y \]

II-2. Equations of rotational motion:

Rigid bodies support moments, in this paragraph we will determine the effects caused by the moments of the external force system computed about an axis perpendicular to the plane of motion (the z-axis) and passing through point P. As shown in the free-body diagram of the i\(^{th}\) particle, \( \vec{F}_i \) represents the resultant external force on the particle, and \( \vec{f}_i \) is the resultant of the internal forces caused by interactions with adjacent particles. If the particle has a mass \( m_i \) and acceleration \( \vec{a}_i \) then

\[ (\vec{M}_p)_i = \vec{r} \times \vec{F}_i + \vec{r} \times \vec{f}_i = m_i \vec{r} \times \vec{a}_i \]

the moments about P can be expressed in terms of the acceleration of point P. In fact if the body has an angular acceleration \( \vec{\alpha} \) and angular velocity \( \vec{\omega} \) then

\[ (\vec{M}_p)_i = m_i \vec{r} \times \vec{a}_i = m_i \vec{r} \times (\vec{a}_p + \vec{\alpha} \times \vec{r} - \vec{\omega}^2 \vec{r}) = m_i [\vec{r} \times \vec{a}_p + \vec{r} \times (\vec{\alpha} \times \vec{r})] \]

Expressing the vectors with cartesian components and carrying out the cross product operations yields

\[ (M_p)_i \hat{k} = m_i \left[ -y (a_p)_x + x (a_p)_y + \alpha r^2 \right] \]

Letting \( m_i \rightarrow dm \) and integrating with respect to the entire mass of the body, we obtain the resultant moment equation

\[ \sum M_p = - \left( \int_m y \; dm \right) (a_p)_x + \left( \int_m x \; dm \right) (a_p)_y + \left( \int_m r^2 \; dm \right) \alpha \]

Here \( \sum M_p \) represents only the moments of the external forces acting on the body about point P. The resultant moment of the internal forces is zero, since for the entire body these forces occur in equal and opposite collinear pairs and thus the moment of each pair of forces about P cancels. On the other hand \( m x_G = \int x \; dm \) \( m y_G = \int y \; dm \) \( I_p = \int r^2 \; dm \). The previous relation reduces
\[ \sum M_p = -m y_G(a_p)_x + m x_G(a_p)_y + I_p \alpha \]

It is possible to reduce this equation to a simpler form if point P coincides with the mass center G for the body. If this is the case, then

\[ \sum M_p = I_G \alpha \]

This rotational equation of motion states that the sum of the moments of all the external forces computed about the body’s mass center G is equal to the product of the moment of inertia of the body about an axis passing through G and the body’s angular acceleration.

One additional thing using the parallel axis theorem we have \( I_p = I_G = m (x_G^2 + y_G^2) \). If we plug this in the previous equation of motion we get

\[ \sum M_p = m y_G(-a_p)_x + y_G \alpha + m x_G((a_p)_y + x_G \alpha) + I_G \alpha \]

but \( \vec{a}_G = \vec{a}_p + \vec{\alpha} \times \vec{r}_G - \vec{w}^2 \vec{r}_G \). carrying out the cross product and equating the respective \( \hat{i} \) and \( \hat{j} \) components yields the two scalar equations

\[ (a_G)_x = (a_p)_x - y_G \alpha - x_G \omega^2 \]
\[ (a_G)_y = (a_p)_y - x_G \alpha - y_G \omega^2 \]

Substituting these results into the previous equation and simplifying gives

\[ \sum M_p = m y_G - (a_G)_x + m x_G(a_G)_y + I_G \alpha \]

This important result indicates that when moments of the external forces shown on the free body diagram are summed about point P, they are equivalent to the sum of the kinetic moments of the components of \( m \vec{a}_G \) about P plus the kinetic moment of \( I_G \alpha \)

II-3. General application of the equations of motion:

To summarize this analysis, three independent scalar equations may be written to describe the general plane of a symmetrical rigid body

\[ \sum F_x = m (a_G)_x \]
\[ \sum F_y = m (a_G)_y \]
\[ \sum M_G = I_G \alpha \text{ or } \sum M_p = -m y_G(a_G)_x + m x_G(a_G)_y + I_G \alpha \]
III. Equations of motion: Translation

When a rigid body undergoes translation, all the particles of the body have the same acceleration, so that \( a_G = a \). Furthermore \( \alpha = 0 \). Application of this and the translational equations of motion will now be discussed for each of the two types of translation.

- **Rectilinear translation:** When the body is subjected to rectilinear translation, all the particles of the body travel along parallel straight line paths. Since \( I_G \alpha = 0 \), the equations of motion reduce to

\[
\begin{align*}
\sum F_x &= m (a_G)_x \\
\sum F_y &= m (a_G)_y \\
\sum M_G &= 0
\end{align*}
\]

In the last equation, if the moments are summed about a point \( A \) different than \( G \) then this equation becomes

\[
\sum M_A = -m y_G (a_G)_x + m x_G (a_G)_y = (ma_G)d
\]

which is the moment of \( m\vec{a}_G \) about \( A \)

- **Curvilinear translation:** When the body is subjected to curvilinear translation, all the particles of the body travel along parallel curved paths. For analysis, it is convenient to use an inertial frame having an origin which coincides with the body’s mass center at the instant considered, and axes which are oriented in the normal and tangential directions to the path of motion. The three scalar equations of motion are then

\[
\begin{align*}
\sum F_n &= m (a_G)_n \\
\sum F_t &= m (a_G)_t \\
\sum M_G &= 0
\end{align*}
\]

In the last equation, if the moments are summed about a point \( B \) different than \( G \) then this equation becomes

\[
\sum M_B = -m e [(a_G)_t] + m h [(a_G)_n]
\]

which is the moment of \( m\vec{a}_G \) about \( A \)
• **Procedure for solving problems:** Kinetic problems involving rigid-body translation can be solved using the following procedure:

  - **Free-body diagram:**
    * Establish the x, y or n, t inertial frame of reference and draw the free-body diagram in order to account for all the external forces and couple moments that act on the body
    * The direction and sense of the acceleration of the body’s mass center \( \vec{a}_G \) should be established
    * Identify the unknowns in the problems

  - **Equations of motion:**
    * Apply the three equations of motion in accordance with the established sign convention
    * To simplify the analysis, the moment equation \( \sum M_G = 0 \) can be replaced by the more general equation about a point P, where point P is usually located at the intersection of the lines of action of as many unknown forces as possible.

  - **Kinematics:**
    * Use kinematics if the velocity and position of the body are to be determined
    * For rectilinear translation with variable acceleration use
      \[
      a_G = \frac{dv_G}{dt}, \quad a_G \, ds_G = v_G \, dv_G, \quad v_G = \frac{ds_G}{dt}
      \]
    * For rectilinear translation with constant acceleration, use
      \[
      v_G = (v_G)_o + a_G \, t, \quad v_G^2 = (v_G)_o^2 + 2a_G[s_G - (s_G)_o],
      \]
      \[
      s_G = (s_G)_o + (v_G)_o \, t + \frac{1}{2} a_G \, t^2
      \]
    * For curvilinear translation, use
      \[
      (v_G)_n = \frac{v_G^2}{\rho}, \quad (a_G)_t = \frac{dv_G}{dt},
      \]
      \[
      (a_G)_t \, ds_G = v_G \, dv_G, \quad (a_G)_t = \rho \, \alpha
      \]
IV. Equations of motion: Rotation about a fixed axis

Consider the rigid body in the figure below which is constrained to rotate in the vertical plane about a fixed axis perpendicular to the page and passing through the pin at O. The angular velocity and angular acceleration are caused by the external force and couple moment system acting on the body. Because the body’s center of mass G moves in a circular path, the acceleration of this point is represented by its tangential and normal components. The free body of diagram of the body is also shown in the same figure.

![Diagram of rigid body](image)

The three equations of motion are

\[ \sum F_n = m (a_G)_n = m\omega^2 r_G \]
\[ \sum F_t = m (a_G)_t = m\alpha r_G \]
\[ \sum M_G = I_G \alpha \]

Often it is convenient to sum the moment about the pin at O in order to eliminate the moment unknown force \( \vec{F}_o \). In this case we have to account for the moment about O of \( m\vec{a}_G \). The moment equation becomes

\[ \sum M_o = m r_G (a_G)_t + I_G \alpha = (I_G + m r_G^2)\alpha \]

From the parallel axis theorem \( I_o = I_G + m r_G^2 \) and therefore the term in parentheses represents the moment of inertia of the body about the fixed axis of rotation passing through O. Consequently we can write the three equations of motion of the body as

\[ \sum F_n = m (a_G)_n = m\omega^2 r_G \]
\[ \sum F_t = m (a_G)_t = m\alpha r_G \]
\[ \sum M_o = I_o \alpha \]

- **Procedure for solving problems:** Kinetic problems involving rotation of a body about a fixed axis can be solved using the following procedure:

  - **Free-body diagram:**
    - Establish the x, y or n, t inertial frame of reference and specify the directions and sense of the acceleration \( a_{Gn}, a_{Gt} \), and the angular acceleration of the body \( \alpha \).
    - Draw the free-body diagram in order to account for all the external forces and couple moments that act on the body.
* Compute the moment of inertia \( I_G \) or \( I_o \)

* Identify the unknowns in the problems

**Equations of motion:**

* Apply the three equations of motion in accordance with the established sign convention

**Kinematics:**

* Use kinematics if a complete solution cannot be obtained strictly from the equations of motion

* If angular acceleration is variable, use

\[
\alpha = \frac{d\omega}{dt}, \quad \alpha \ d\theta = \omega \ d\theta, \quad \omega = \frac{d\theta}{dt}
\]

* If angular acceleration is constant, use

\[
\omega = \omega_o + \alpha_c \ t, \quad \omega^2 = \omega_o^2 + 2\alpha_c [\theta - \theta_o],
\]

\[
\theta = \theta_o + \omega_o \ t + \frac{1}{2} \alpha_c \ t^2
\]

**V. Equations of motion: General plane motion**

The rigid body in the figure below is subjected to a general plane motion caused by externally applied force and couple moment system. If an external \( x, y \), inertial coordinate system is used the three equations of motion are

\[
\sum F_x = m \ (a_G)_x
\]

\[
\sum F_y = m \ (a_G)_y
\]

\[
\sum M_G = I_G \ \alpha
\]

In some problems it may be convenient to sum moments about some point \( P \) other than \( G \). This is usually done in order to eliminate unknown forces from the moment summation. When used in this more general sense, the three equations of motion become

\[
\sum F_x = m \ (a_G)_x
\]

\[
\sum F_y = m \ (a_G)_y
\]

\[
\sum M_p = \sum (M_k)_p
\]

Here \( \sum (M_k)_p \) represents the moment sum of \( I_G \ \alpha \) and \( m \ \vec{a}_G \) (or its components) about \( P \).

**Frictional rolling problems:** In addition to the three equations of motion for general plane motion, frictional rolling problems (involving e.g., wheels, disks, cylinders, or balls) often require an extra equation due to the presence of the extra unknown representing the frictional force. There are two cases

- **No slipping:** In this case we have the extra equation

\[
a_G = r \ \alpha
\]

Note that when the solution is obtained, the assumption of no slipping must be checked (i.e., verify that \( F < \mu_s \ N \)) otherwise it is necessary to rework the problem under the assumption of slipping.

- **Slipping:** Here \( \alpha \) and \( \vec{a}_G \) are independent of each other so instead we relate the magnitude of the frictional force \( \vec{F} \) to the magnitude of the normal force \( \vec{N} \) using the coefficient of kinetic friction \( \mu_k \) and obtain the extra equation
\[ F = \mu_k \, N \]

- **Procedure for solving problems:** Kinetic problems involving general motion of a body can be solved using the following procedure:

  - **Free-body diagram:**
    * Establish the x, y inertial frame of reference and draw the free body diagram in order to account for all the external forces and couple moments that act on the body
    * The directions and sense of the acceleration \( a_{ Gn} \), and the angular acceleration of the body \( \alpha \) should be established
    * Compute the moment of inertia \( I_{ G} \)
    * Identify the unknowns in the problems

  - **Equations of motion:**
    * Apply the three equations of motion in accordance with the established sign convention
    * When friction os present, there is a possibility for motion wit h no slipping or tripping. Each possibility for motion should be considered

  - **Kinematics:**
    * Use kinematics if a complete solution cannot be obtained strictly from the equations of motion
    * If the body’s motion is constrained due to its supports, additional equations may be obtained by \( \vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \), which relate the acceleration of any two points A and B on the body.
    * When a wheel, disk, cylinder, or ball rolls without slipping then \( a_G = r \, \alpha \)