**I. Kinetic Energy:**

Consider a rigid body subjected to general planar motion. The object can be considered as a collection of particle points scattered over the entire volume of the object each of mass \( m_i \), velocity \( \vec{v}_i \).

![Diagram of rigid body with particle points and velocity vectors](image)

The kinetic energy of the rigid body \((KE)\) is the sum of the kinetic energies of all its particles

\[
KE = \sum KE_i = \frac{1}{2} m_i v_i^2
\]

This equation may also be written in terms of the velocity of point \( P \). If the body has an angular velocity \( \vec{\omega} \) then

\[
\vec{v}_i = \vec{v}_P + \vec{\omega} \times \vec{r}_{i/P} = [(v_P)_x - w y_i]\hat{i} + [(v_P)_y - w x_i]\hat{j}
\]

Substituting into the equation of kinetic energy yields

\[
KE = \frac{1}{2} \left( \sum m_i \right) v_P^2 + \frac{1}{2} I_p \omega^2 - \left( \sum m_i \right) (v_P)_x \omega y_G + \left( \sum m_i \right) (v_P)_y \omega x_G
\]

As a special case, if point \( P \) coincides with the center of mass \( G \) for the body then

\[
KE = \frac{1}{2} M v_G^2 + \frac{1}{2} I_G \omega^2
\]

Here \( M \) is the mass of the object, \( \omega \) is the body’s angular velocity, and \( I_G \) is the body’s moment of inertia about an axis perpendicular to the plane and passing through the center of mass.

- **Translation:** When the rigid body is subjected to either rectilinear or curvilinear translation, the kinetic energy due to rotation is zero since \( \omega = 0 \). The previous expression reduces to

\[
KE = \frac{1}{2} M v_G^2
\]
• **Rotation about a fixed axis:** When the rigid body is rotating about a fixed axis passing through point \( O \), the body has both translational and rotational kinetic energy, but since \( v_G = r_G \omega \), The previous expression reduces to

\[
KE = \frac{1}{2} M r_G \omega^2 + \frac{1}{2} I_G \omega^2 = \frac{1}{2} (I_G + m r_G^2) \omega^2 = \frac{1}{2} I_o \omega^2
\]

• **General plane motion:** When the rigid body is subjected to general plane motion, it has an angular velocity \( \omega \) and its mass center has a velocity \( v_G \); the kinetic energy is

\[
KE = \frac{1}{2} M v_G^2 + \frac{1}{2} I_G \omega^2
\]

• Because energy is a scalar quantity, the total kinetic energy of a system of connected rigid bodies is the sum of the kinetic energies of all its moving parts.
II. The work of a force:

- **Work of a variable force**: If an external force acts on a rigid body, the work done by the force when it moves along the path $s$ is defined as

  $$Work = \int_s F \cos(\theta) \, ds$$

  where $\theta$ is the angle between the tail of the force vector and the differential displacement.

- **Work of a constant force**: If an external force $\vec{F}_c$ acts on a rigid body, and maintains a constant magnitude $F_c$ and constant direction $\theta$ the work becomes

  $$Work_{F_c} = F_c \cos(\theta) \, s$$

- **Work of a weight**: The weight of a body does work only when the body’s mass center $G$ undergoes a vertical displacement $\Delta y$. If the displacement is upward, the work is negative

  $$Work_W = -W \, \Delta y$$
• **Work of spring force:** If a linearly elastic spring is attached to a body, the spring force $F = ks$ acting on the body does work when the spring either stretches or compresses from $s_1$ to further position $s_2$. In both cases the work is negative (since body’s displacement is in opposite direction of force).

$$Work_S = -\frac{1}{2}k(s_2^2 - s_1^2)$$ where $|s_2| > |s_1|$

• **forces that do no work:** There are some external forces that do no work when the body is displaced. These forces can act either at fixed points on the body or they can have a direction perpendicular to their displacement. e.g., reactions at a pin support about which a body rotates or the weight of the body when the center of gravity of the body moves in a horizontal plane. A rolling resistance force $F_r$ acting on a round body as it rolls without slipping over a rough surface also does no work. This is because, during any instant of time $dt$, $F_r$ acts at a point on the body which has zero velocity (Instantaneous center of zero velocity, IC), and so the work done by the force on the point is zero. Since $F_r$ contacts successive points for only an instant, the work of $F_r$ will be zero.
III. The work of a couple:

When a body is subjected to a couple undergoes general plane motion, the two couple forces do work only when the body undergoes a rotation. To show this, consider the body in the figure below, which is subjected to a couple moment perpendicular to the plane motion of the object. For simplicity assume that the couple force system is applied at $\vec{r}_1$ and $\vec{r}_2$ from a certain reference point $O$ on the body. we have the following

\[ d\vec{s}_1 = d\vec{s}_o + d\vec{r}_1 = d\vec{s}_o + \vec{\omega}dt \times \vec{r}_1 \]
\[ d\vec{s}_2 = d\vec{s}_o + d\vec{r}_2 = d\vec{s}_o + \vec{\omega}dt \times \vec{r}_2 \]
\[ d\text{Work} = \vec{F} \cdot d\vec{s}_1 - \vec{F} \cdot d\vec{s}_2 = \vec{F} \cdot (\vec{\omega}dt \times (\vec{r}_1 - \vec{r}_2)) = \vec{\omega}dt \cdot ((\vec{r}_1 - \vec{r}_2) \times \vec{F}) = M_o \vec{\omega}dt = M d\theta \]
\[ \text{Work} = \int_{\theta_1}^{\theta_2} M d\theta \]

If the couple moment has a constant magnitude then
\[ \text{Work} = M (\theta_2 - \theta_1) \]

Here the work is positive provided $M$ and $(\theta_2 - \theta_1)$ are in the same direction.
IV. The Principle of work and energy:

By applying the principle of work and energy developed in chapter 3 to each of the particles of a rigid body and adding the results algebraically, since energy is a scalar, the principle of work and energy for a rigid body becomes

\[ KE_1 + \sum Works = KE_2 \]

This equations states that the body’s initial translational and rotational kinetic energy, plus the work done by all the external forces and couple moments acting on the body as the body moves from its initial to its final position, is equal to the body’s final translational and rotational kinetic energy. Note the work of the body’s internal forces does not have to be considered since the body is rigid. These forces occur is equal but opposite collinear pairs, so that when the body moves, the work of one force cancels that of its counterpart. Furthermore since the body is rigid, no relative movement between these forces occurs, so that no internal work is done.

When several rigid bodies are pin connected, connected by inextensible cables, or in mesh with one another. The previous equation can be applied to the entire system of connected bodies. In all these cases the internal forces, which hold the various members together, do no work and hence are eliminated from the analysis.

IV-1. Procedure for solving problems:

The principle of work and energy is used to solve kinetic problems that involve velocity, force and displacement, since these terms are involved in the formulation. We use the following procedure:

- **Work (free-body diagram)**
  - Draw a free-body diagram of the body when it is located at an intermediate point along the path in order to account for all the forces and couple moments which do work on the body as it moves along the path
  - A force does work when it moves through a displacement in the direction of the force
  - Forces that are function of displacement must be integrated to obtain the work
  - Since algebraic addition of the work terms is required, it is important that the proper sign of each terms be specified. Specifically work is positive when the force (couple moment) is in the same direction as its displacement (rotation); otherwise it is negative

- **Principle of work and energy**
  - Apply the principle of work and energy \( KE_1 + \sum Works = KE_2 \). Since this is a scalar equation, it can be used to solve for only one unknown when it is applied to a single rigid body
V. Conservation of energy:

When a force system acting on a rigid body consists only of conservative forces, the conservation of energy theorem can be used to solve a problem which otherwise would be solved using the principle of work and energy. This theorem is easier to apply since the work of a conservative force is independent of the path and depends only on the initial and final positions of the body. Recall that the work of a conservative force may be expressed as the difference in the body’s potential energy measured from an arbitrarily selected reference.

\[ U_{1\rightarrow 2} = Work_{c2\rightarrow 1} = -Work_{c1\rightarrow 2} \]

where \( U_{1\rightarrow 2} = U_2 - U_1 \): the change in the potential energy between positions 1 and 2. \( Work_{c2\rightarrow 1} \) is the work done by the conservative force in taking the object from position 2 to position 1.

- **Gravitational potential energy:** Since the total weight of a body can be considered concentrated at its center of gravity, the gravitational potential energy of the body is determined by knowing the height of the body’s center of gravity above or below a horizontal reference line (datum)

\[ U_g = W \ y_G \]

- **Elastic potential energy** The force developed by an elastic spring is also a conservative force. The elastic potential energy which a spring imparts to an attached body when the spring is elongated or compressed from an initial undeformed position \( s = 0 \) to a final position \( S \) is

\[ U_e = \frac{1}{2} ks^2 \]

- **Conservation of energy:** In general, if a body is subjected to both gravitational and elastic forces, the total potential energy is expressed as \( U = U_g + U_e \). Now realizing that the work of the conservative forces can be written as a difference in potential energies we can rewrite the principle of work and energy as

\[ KE_1 + U_1 + (\sum Work_{c1\rightarrow 2})_{noncons} = KE_2 + U_2 \]
Here \( \sum \text{Work}_{1\rightarrow2}^{\text{noncons}} \) represents the work of the non-conservative forces such as friction. If this term is zero then

\[
KE_1 + U_1 = KE_2 + U_2
\]

This equation is referred to as the conservation of mechanical energy. It states that the sum of the potential and kinetic energies of the body remains constant when the body moves from one position to another. It also applies to a system of smooth, pin-connected rigid bodies, bodies connected by inextensible cords, and bodies in mesh with other bodies. In all these cases the forces acting at the points of contact are eliminated from the analysis, since they occur in equal but opposite collinear pairs and each pair of forces move through an equal distance when the system undergoes a displacement.

It is important to remember that only problems involving conservative force systems may be solved using the previous equation. As stated before, friction or other drag-resistant forces, which depend of velocity or acceleration, are non-conservative. The work of such forces is transformed into thermal energy used to heat up the surfaces of contact, and consequently this energy is dissipated into the surrounding and may not be recovered. Therefore, problems involving frictional forces can be solved by using either the principle of work and energy or the equations of motion.