

**LAB 1*****FOURIER TRANSFORM  
&  
AMPLITUDE MODULATION*****OBJECTIVES**

- Derive the Fourier transform of an aperiodic signal to find its frequency content.
- Apply Fourier transform properties for Amplitude Modulation in communication systems.

**INTRODUCTION****Fourier Transform**

The Fourier transform is a mathematical operation that finds the spectrum  $X(\omega)$  of an aperiodic continuous signal  $x(t)$  as shown by Equation (1).

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad (1)$$

$X(\omega)$  is a continuous signal in the frequency domain that represent the frequency content of the signal  $x(t)$ . Usually,  $X(\omega)$  is complex signal where the magnitude and phase represent the strength and the phase shift (time delay) of the frequency components, respectively. For example, the signal  $x(t) = \text{rect}(t/\tau)$  has the Fourier transform  $X(\omega) = \tau \text{sinc}(\omega\tau/2)$ , shown in Fig. 1.

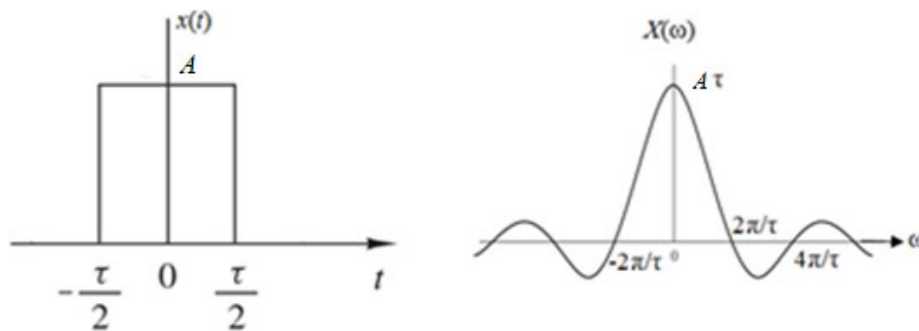


Fig. 1:  $x(t) = A \text{rect}(t/\tau)$  has the Fourier transform  $X(\omega) = A\tau \text{sinc}(\omega\tau/2)$

The spectrum of the signal  $x(t)$  is continuous that ranges from DC (0 Hz) to infinite frequency. The negative frequency in the  $\omega$ -axis in the above figure does not exist in reality. The negative frequencies are there because of the mathematical nature of the Fourier transform. It is obvious that the energy of the signal is concentrated in the main lobe (0 to  $2\pi/\tau$  rad/sec) as shown by the spectrum. Higher frequencies are represented in the smaller side lobes due to the presence of sharp edges in the signal  $x(t)$ . Sharp edges in a signal mean rapid change in the signal. Elimination of the high frequencies will result in rounding the edge of the original signal. This

phenomenon can be observed in oscilloscopes with limited frequency bandwidth. A square wave with sharp edges (contain high frequency components) will be displayed as a square wave with rounded edges because these oscilloscopes are unable to display all high frequency components of the square wave.

Observe the relationship between the time width of the pulse  $x(t)$ , and the frequency width of the main loop in the aspectrum  $X(\omega)$ . The narrower the time interval (small  $\tau$ ), the larger the spectral band of the main loop (wider main loop). This makes sense because pulse with narrower width means high rate of change between 0 V and A V in a short period of time  $\tau$ . High rate of changes in the time domain means the signal has high frequency component, which usually results in wider main loop and many small loops. In communication systems, if the internet speed is 1 MBPS (Mega bit per second), then the maximum rate of transmission is 1,000,000 (million) pulses per second, meaning that the pulse width is approximately one microsecond. For transmission at a higher rate, the pulse width must be narrower. Narrower pulse means the spectrum will be larger. Larger spectrum requires larger channel bandwidth which is costly. Effect of narrow pulse and its spectrum is also applicable in mechanical systems. Narrower pulses mean fast movement in the mechanical system. Fast mechanical movement will cause higher vibration (turbulence) on the system.

### Windowing in Data Truncation

Windowing is the process of multiplying the signal  $x(t)$  by a rectangular pulse of specific duration to select specific set of data in the signal  $x(t)$ . Windowing is used in data compression and data processing by segmenting the data to multiple intervals of small durations. Unfortunately, the negative consequence of windowing is the distortion of the signal spectrum. This distortion takes the form of spectrum spreading, spectrum leakage, and spectrum resolution degradation. For example, a sinusoidal signal that exists over a long period of time as shown in Fig.2a has a Fourier transform as shown in Fig. 2b. The spectrum is an impulse at  $\omega_0$  which is the frequency of the sinusoidal signal  $x(t)$ .

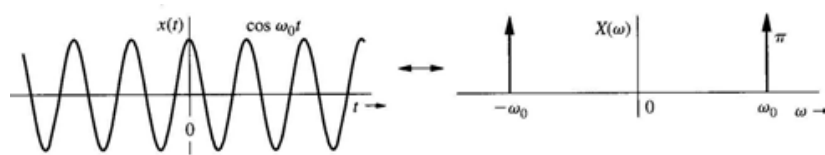


Fig. 2a

Fig. 2b

Now, truncating the signal  $x(t)$  by multiplying it by a rectangular window (Fig 3a), produce the truncated time limited signal  $x_w(t)$  as shown in Fig. 3b and finally, the Fourier transform can be displayed as shown in Fig. 3c.

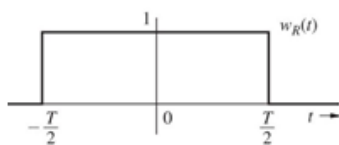


Fig. 3a

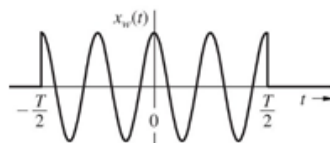


Fig. 3b

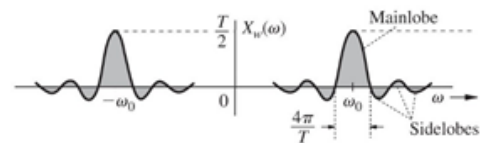


Fig. 3c

In Fig. 3c the spectrum of the truncated sinusoidal  $x_w(t)$  is more spread when compared to the impulse in Fig. 2b. In the main loop, there is one frequency  $\omega_0$  with the highest amplitude; and other frequencies adjacent to  $\omega_0$  with smaller amplitudes. Also, the spectrum has leakage shown in the smaller side loops. The wider the window in the time domain the smaller is the spread of the main loop in the frequency domain. Now imagine if the signal  $x(t)$  consists of two tones with frequencies  $\omega_0$  and  $\omega_0+2\pi 10$  rad/sec. If we truncate the complex signal by a rectangular window of width less than 0.1 seconds, then the main loops of the two frequencies will overlap, and it will be difficult to discriminate the two frequencies from each other. Using short window deteriorates the frequency resolution of the signal spectrum. Intuitively, short time interval of the signal does not give us enough information about the signal to resolve all frequency components. There is an inverse relationship between the length of the signal in the time domain (window) and the signal spectrum resolution in the frequency domain.

The rule of thumb is that the longer the window, the better the spectral resolution. For example, the spectral resolution of a signal windowed by a 0.1 second window will have frequency resolution equals 10 Hz, while the spectral resolution of a signal windowed by 1 second window will have frequency resolution equals 1 Hz. The drawback of long window is that the Fourier transform of the signal will not show when the different frequency components start and end during the window time interval.

Another example, a complex signal of two frequency components 10 Hz and 11 Hz with 2 second widow truncation will provide a Fourier transform with 0.5 Hz frequency resolution. The frequency resolution of the 2 seconds signal allows us to discriminate between 10 Hz and 11 Hz. However, we will not be able to tell at what interval of the 2 seconds the 11 Hz component exists. In short, longer window gives us better frequency resolution but poor time resolution.

Analytically, a signal  $x(t)$  consist of three sinusoidal as expressed by Equation (2) is shown in Fig. 4.

$$x(t) = 2\sin(2\pi 2t)[\text{rect}(t-2/4)] + \sin(2\pi 5t)[\text{rect}(t-1.5/2)] + 0.5 \sin(2\pi 7t)[\text{rect}(t-2.5/4)] \quad \text{<--> (2)}$$

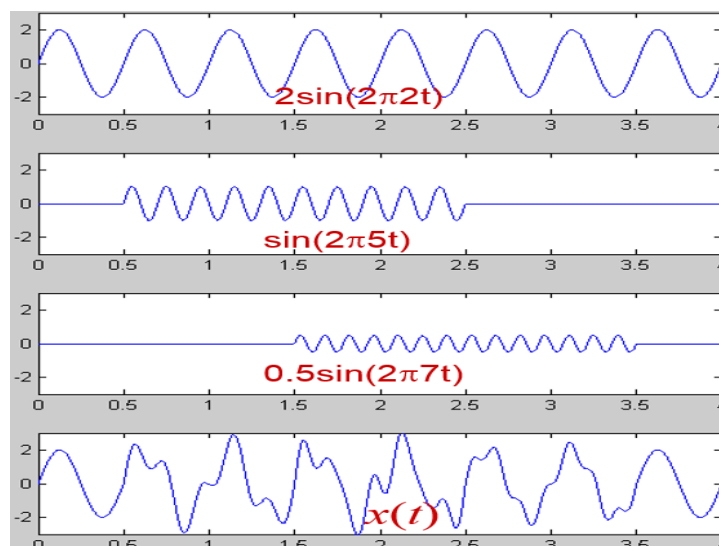


Fig. 4

The Fourier transform of the signal  $x(t)$  for time length of 4 second is shown in Fig. 5. It is obvious that the signal  $x(t)$  has three frequency components 2, 5, and 7 Hz. The 2 Hz frequency component has the highest amplitude or strength, and the 7 Hz frequency component has the lowest. However, the Fourier transform does not show when the 7 Hz sinusoidal starts and when it ends.

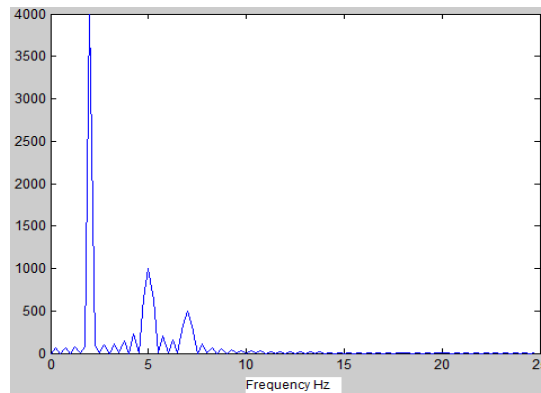


Fig. 5

In order to obtain a better time resolution to show when the 7Hz frequency signal starts, a Short Time Fourier Transform (STFT) can be used for every one second, this is equivalent to windowing with a window of length 1 second. The STFT for the first one second duration of the signal is shown in Fig. 6.

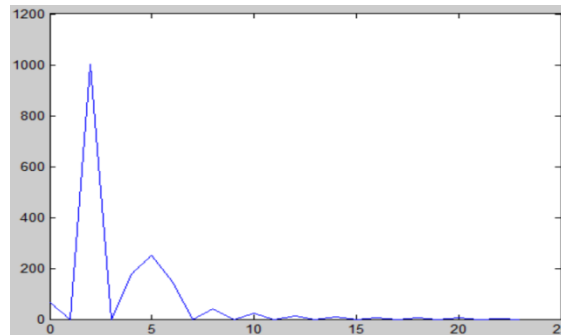


Fig. 6: STFT

From Fig. 6, it is obvious that during the first second only sinusoidal 2 and 5 Hz exist, and the 7 Hz does not exist, which agrees with the signal in the time domain. The advantage of STFT (windowing with short time interval) is that it gives us better time resolution with the disadvantage of poor frequency resolution. Comparing Fig. 5 with Fig. 6, it can be seen that the main loops in Fig. 5 are narrower which indicate pure sinusoidal frequencies of 2Hz and 5 Hz.

If we take the Fourier transform of the signal  $x(t)$  for the time interval from 1 second to 2 second, then the Fourier transform is shown in Fig. 7.

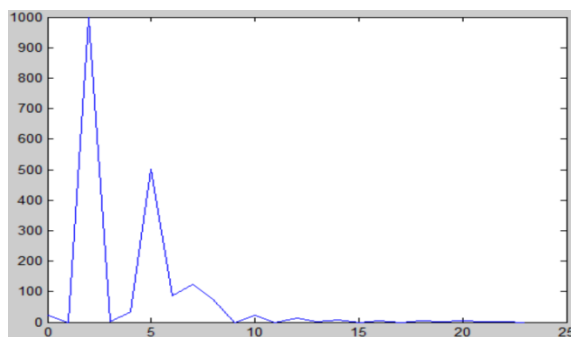


Fig. 7

In Fig. 7, the signal during this time has three frequency components 2, 5, and 7 Hz. Note that the peaks of the 5Hz and 7Hz are not clearly discriminated as it was in Fig. 5. The poor discrimination between the 5 Hz and 7 Hz frequency components is due to the 1 Hz frequency resolution of the Fourier transform. The 1 Hz frequency resolution causes the peak of the 7 Hz not to be obvious. The 1 Hz resolution is the consequence of the one second window.

### Sampling and the Discrete Fourier Transform

The Fourier transform is not generally calculated by the integral shown in Equation (1). The discrete mathematical Fast Fourier Transform (FFT) is more appropriate for discrete signal as represented by Equation (3).

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \quad (3)$$

Here,  $x[n]$  is the samples of the continuous signal  $x(t)$ . The number of samples  $N$  is determined by the sampling rate and the length of the signal  $x(t)$  (window). For example, if the sampling rate is 8000 samples per second, then a signal  $x(t)$  windowed by 2 seconds length will generate a discrete signal  $x[n]$  of length 16,000 samples. Note, the sampling rate must be equal or larger than twice the highest frequency in the signal  $x(t)$ .

The FFT of discrete signal will generate a Fourier transform of discrete frequencies. The frequency resolution will be the reciprocal of the time length of the signal (window). For example, a signal of length 2 second will have a discrete Fourier transform with frequency resolution equal 0.5 Hz (1/2 seconds).

The term  $k$  in Equation (3) represents the harmonic frequency. For example, for  $k=3$ ,  $X[3]$  represents the strength of the third harmonic. If the frequency resolution is 0.5 Hz then  $X[3]$  is the strength of the 1.5 Hz. For  $k=0$ ,  $X[0]$  is the dc component in the signal. The highest frequency component in the signal  $x(t)$  will be half the sampling rate.

To summarize, the discrete Fourier transform can be linked to the continuous Fourier transform by the rules below:

1. The sampling rate must be larger than twice the bandwidth of the signal.
2. The frequency resolution of the FFT equal the reciprocal of the signal length (window size)
3. The number of samples will produce frequency components that is equal to the half the number of samples
4. The lowest frequency is DC ( $k = 0$ ), and the highest frequency is half of the sampling rate.

### **Modulation in Signal Transmission**

Modulation is used to shift the spectrum of a signal to a different spectrum band. This can be accomplished by multiplying the signal  $x(t)$  by a sinusoidal signal of high frequency.

$$s(t) = x(t) \sin(2\pi f_m t)$$

The spectrum of the signal  $x(t)$  will be centered around  $f_m$  instead of 0 Hz. For example, a speech signal with frequency band from 20 Hz to 20 kHz modulated with 1 MHz carrier frequency will shift the spectrum  $X(\omega)$  to be centered around 1 MHz. The frequency,  $f_m$  of the modulator signal is called the carrier frequency.

In communication systems, modulation is used in wireless transmission for two reasons. The first reason is to minimize the length of the antenna used for transmission. The antenna's length should be at least half the longest wavelength  $\lambda$  present in the transmitted signal. The relationship between frequency and wavelength is  $\lambda = c/f$ , where  $c$  is the speed of light. Therefore, a 50 Hz frequency component has wavelength equal  $\lambda = 6$  million meter. In order to transmit a 50 Hz component of the speech signal wirelessly, an antenna with length equal 3 million meter ( $\lambda/2$ ) is required. Antenna with this length is not possible. Shifting the spectrum of the speech to 100 MHz will require antenna of length 1.5 meter, which is more practical.

The second reason for modulation is to shift the spectrums of different signals to different frequency bands so that the receiver can separate the received signals by using band-pass filters whose central frequencies equal the corresponding carrier frequencies. Amplitude modulation is common in broadcasting (AM radio and TV) as well as in point-to-point communications (walkie-talkies). The signal information is used to alter the amplitude of the carrier signal. Usually, the frequency of the carrier signal is very high (10 MHz) in comparison to the modulated signal (voice between 20 Hz to 4000Hz).

A message signal  $m(t)$ , a carrier signal  $c(t)$ , and a modulated signal  $u(t)$  are plotted in Fig. 8 at the upper, middle, and lower panels, respectively. The mathematical expression of these signals are shown below:

$$\begin{aligned} m(t) &= \cos(2\pi 2t) + 2\cos(2\pi 3t) + 2 \\ c(t) &= \sin(2\pi 30t) \\ u(t) &= [\cos(2\pi 2t) + 2\cos(2\pi 3t) + 2] \sin(2\pi 30t) \end{aligned}$$

The MATLAB code for the above equations and the plot is

```
>> t=0:0.00001:1;  
>> mt=2*sin(2*pi*2*t)+ sin(2*pi*3*t)+2;  
>> ct=sin(2*pi*30*t);  
>> ut = mt.*ct;  
>> subplot(3,1,1)plot(t,mt)  
>> subplot(3,1,2)plot(t,ct)  
>> subplot(3,1,3)plot(t,ut)
```

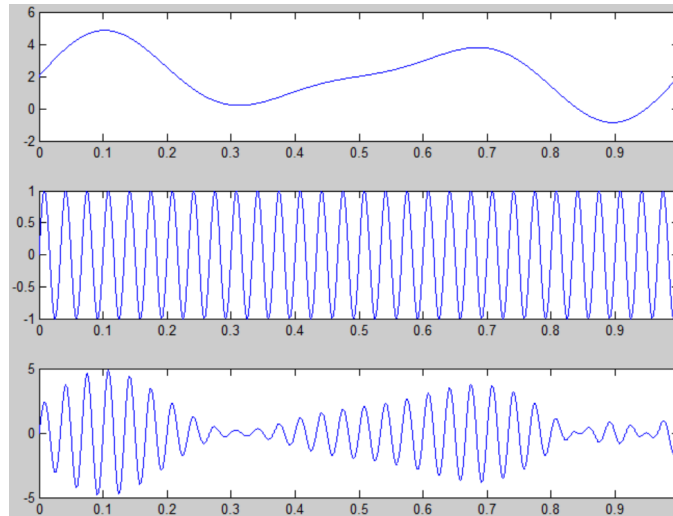


Fig. 8

The envelope of the lower panel in Fig. 8 is similar to the signal in upper panel. In this lab, amplitude modulation will be studied where the modulator is a complex sinusoidal signal that consists of three low frequencies (dc-0Hz, 5Hz, and 10Hz) with a higher carrier frequency, 40 Hz. We will investigate how modulation shifts the spectrum of the signal to be centered on the frequency of the carrier.

**PRELAB EXERCISES**

1. Use the Fourier transform integral to show that the Fourier transform of the signal  $x(t) = A \text{rect}(t/\tau)$  shown below is  $X(\omega) = A \tau \text{sinc}(\omega\tau/2)$

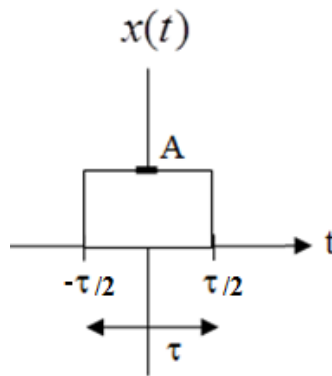


Fig. 9

Fig. 9

**FFT of a Rectangular Window of Different Widths.**

Write and run the Matlab code below that creates a pulse of two seconds time width.

```
clear all
A = 5;
taw = 2;
SampRate = 1000;
t=-2:1/SampRate:2;
xt =A*((t>=-taw/2) & (t<=taw/2));
plot(t, xt)
axis([-2 2 -1 6])
```

The Fourier transform of the pulse  $x(t) = A \text{rect}(t/2)$  can be found in MATLAB by using the fft function. Add the two lines of code below to the above code:

```
Xw=fft(xt);
Xw=abs(fftshift(Xw));
```

The MATLAB function fftshift centers the spectrum  $X(\omega)$  of the signal  $x(t)$  around zero. To plot the Fourier transform  $X(\omega)$ , add the next 4 lines of code to the above code.

```
FreqResolution = 1/(max(t)-min(t));
Fmax = SampRate/2;
FreqAxis=-Fmax:FreqResolution:Fmax;
plot(FreqAxis, Xw)
```

In order to see the detail of the plot around zero frequency, use the zooming tool in Matlab to zoom in



2. Write a code to plot a pulse of width 5 seconds and plot its Fourier transform on the appropriate axis. Draw the plots with your hand inside the box below.

3. What is the mathematical relationship between the pulse width and the main lobe width?

4. Find the first three values of  $\omega$  for which the spectrum  $|X(\omega)|$  equals zero for the 5 seconds pulse by using  $X(\omega)$  and from the plot in task 2.

**Call the instructor to verify the plots.**

**Attendant Signature:** \_\_\_\_\_

**LAB EXERCISES****Discrete Fourier Transform**

1. What is the fundamental frequency of the signal  $x(t)$  shown below? \_\_\_\_\_

$$x(t) = 1 + 0.25\cos(4\pi t - \pi/4) + 0.5\cos(8\pi t)$$

2. The code below will sample the signal  $x(t)$  at a rate of  $f_s = 10$  samples/sec that is above the Nyquist rate to obtain the discrete signal  $x[n]$ .

```
To=1;
fs=10;
ts=0:1/fs:( To -1/fs);
xn=1+0.25*cos(2*pi*2*ts-pi/4)+0.5*cos(2*pi*4*ts);
stem(ts,xn);    %plot xn using stem function
```

3. Use the `fft` and the `fftshift` MATLAB functions to find the DFT coefficients of the discrete signal  $x[n]$ . Add the code line below to above code.

```
Fx=fftshift( (fft(xn)) );
```

The `fftshift` function is to center the Fourier transform magnitude around  $\omega = 0$ .

4. Add the code below to previous codes to plot the magnitude and the phase of the discrete spectrum  $X(k\omega_0)$  which is the same as  $X[k]$  mentioned in the introduction of this lab.

```
stem([-fs/2: 1/To: fs/2-1/To],abs(Fx));
figure
stem([-fs/2: 1/To: fs/2-1/To],angle(Fx).*(abs(Fx)>0.5));
```

Verify the frequency components and their corresponding phases that are present in the signal  $x(t)$  from  $X[k]$ .

5. What is the spectrum resolution  $f_o$  (frequency sampling interval) of the discrete spectrum  $X[k]$ ? \_\_\_\_\_

6. Recover the discrete signal  $x[n]$  from its discrete spectrum  $X[k]$  using the inverse Fourier transform MATLAB built-in function `ifft`.

```
xr=ifft(ifftshift(Fx));
```

Do the recovered samples  $x[n]$  from the inverse discrete Fourier transform agree with the original samples  $x[n]$ ?

\_\_\_\_\_

**Effect of Data Truncation on Spectrum Resolution of the Data**

Change the frequency of the first cosine term to 2.5 Hz instead of 2 Hz.

$$x(t) = 1 + 0.25\cos(2\pi 2.5t - \pi/4) + 0.5\cos(2\pi 4t)$$

7. What is the fundamental frequency  $f_o$  of the above signal? \_\_\_\_\_
8. Repeat steps of Discrete Fourier Transform. Change the frequency from 2Hz to 2.5Hz in your code.
9. Why does the spectrum  $X[k]$  of the sampled signal  $x_n$  show frequency components that are not present in the continuous signal  $x(t)$ , and does not show the 2.5Hz frequency which is present in the  $x(t)$ ?

10. If we sample the signal  $x(t)$  for 2 seconds (window size is 2 seconds), then what is the resolution of the spectrum of the sampled signal  $x[n]$ ? Change T0 to 2 second in the above code and plot the spectrum. Does the spectrum reflect the presence of 2.5 Hz and the 4 Hz frequencies?

**Amplitude Modulation of a Signal**

11. Write the function AmPlot shown below. This function will be used later to plot the three signals (the modulator, the carrier, and the modulated signals) passed to the function.

**MATLAB Code for AmPlot function**

```

function AmPlot(idx,m,c,u,rng)
% idx: x index (it can represent time or frequency)
% m: modulating signal
% c: carrier signal
% u: modulated signal
% rng: range of x axis to plot (optional)
if nargin < 5; rng = 1; end % default value for rng
figure;
subplot(3,1,1);
plot(idx,m); grid on;
axis([rng*min(idx) rng*max(idx) min(m)-0.1 max(m)+0.1]);
subplot(3,1,2);
plot(idx,c); grid on;
axis([rng*min(idx) rng*max(idx) min(c)-0.1 max(c)+0.1]);
subplot(3,1,3);
plot(idx,u); grid on;
axis([rng*min(idx) rng*max(idx) min(u)-0.1 max(u)+0.1]);

```

12. How does the *rng* parameter affect the range of the *x*-axis?

13. For the following steps use sampling rate of 44100 samples/sec. Generate a message signal  $m(t)$  that consist of two frequencies 400 Hz, 700 Hz and a DC.

$$m(t) = 1 + 0.25\cos(2\pi 400t) + 0.5\cos(2\pi 700t)$$

14. Generate the carrier signal  $c(t)$  with a carrier frequency 5000 Hz

$$c(t) = \cos(2\pi 5000t)$$

15. Create a MATLAB script M-File and call it FourierSpectrum. The M-file FourierSpectrum will generate the message signal  $m(t)$ , the carrier signal  $c(t)$ , and then the modulated signal  $u(t) = m(t)*c(t)$ . It will also call the function AmPlot to plot the  $m(t)$ ,  $c(t)$ ,  $u(t)$ ,  $M[k]$ ,  $C[k]$ , and  $U[k]$ .

**MATLAB Code for FourierSpectrum M-File**

```

fs = 44100;
len = 2; % length of signals in seconds
t = (-len/2):1/fs:(len/2); % time index
% generate the message signal m(t)
f1 = 400; f2 = 700;
m = 1 + 0.25*cos(2*pi*f1*t) + 0.5*cos(2*pi*f2*t);
% generate the carrier signal c(t)
fc = 5000; % carrier frequency
c = cos(2*pi*fc*t);
% Modulate the signal
u = m.*c;
% Let's plot them in time to see how they look like
AmPlot(t,m,c,u,5/f1);
% Calculate the spectra
M = abs(fftshift(fft(m)));
C = abs(fftshift(fft(c)));
U = abs(fftshift(fft(u)));
% Make the frequency index for plotting
f = (-fs/2):(1/len):(fs/2);
% Now let's plot them in frequency
AmPlot(f,M,C,U,3*fc/fs);

```

Verify the frequency components in the signal  $m(t)$  from its spectrum.

16. Which frequency component has the highest amplitude? \_\_\_\_\_
17. Use the Matlab function “sound( )” to play the message signal  $m(t)$ , the carrier  $u(t)$  and the modulated signal  $u(t)$ . Can you hear the DC in the message signal  $m(t)$ ? Explain. Is the pitch of  $u(t)$  closer to the carrier or to the message signal?
18. Change the frequency of the carrier to 20,000 Hz and try to listen to the modulated signal  $u(t)$  again. Can you hear the modulated signal  $u(t)$ ? If not, then imagine hearing a modulated message signal with a carrier frequency of 100 MHz.

**NO LAB REPORT FOR THIS LAB**