ENGR 4333/5333: Digital Signal Processing

Discrete-Time Fourier Analysis

Chapter 6

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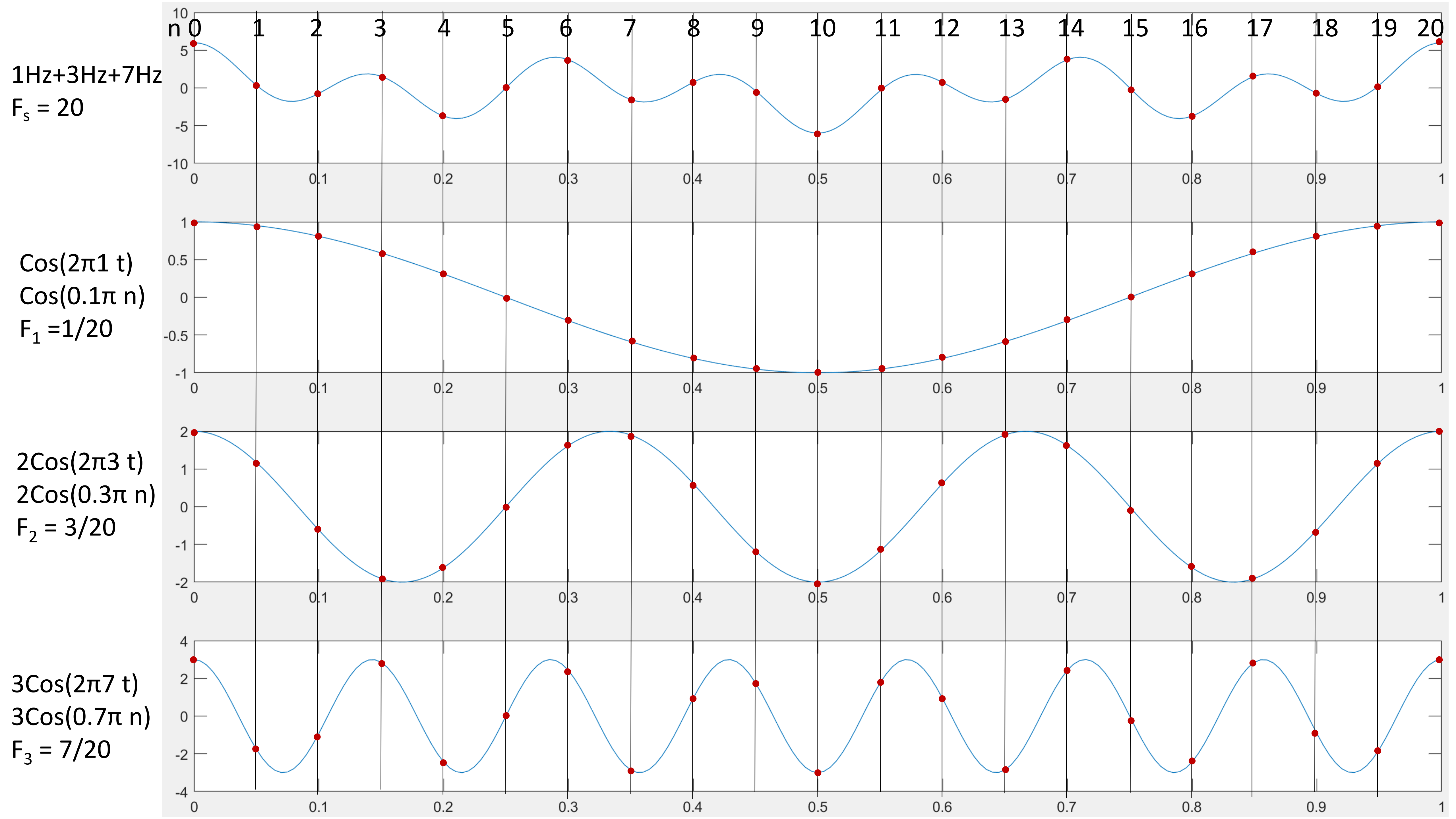
Outline

- Discrete Time Fourier Transform (DTFT)
- Properties of the DTFT
- LTID System Analysis by the DTFT
- Connection between the DTFT and CTFT
- Digital Processing of Analog Signals
- Digital Resampling: A Frequency Domain Perspective
- Generalization of the DTFT to the z-Transform



n 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
1Hz+3Hz+7Hz
F$_s$ = 20
Cos(2π1 t)
Cos(0.1π n)
F$_1$ =1/20
2Cos(2π3 t)
2Cos(0.3π n)
F$_2$ = 3/20
3Cos(2π7 t)
3Cos(0.7π n)
F$_3$ = 7/20
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

The Discrete-Time Fourier Transform (DTFT)

DTFT is a mathematical tool to represent an aperiodic discrete-time signal in terms of its frequency components (*analysis equation*), so it shows the spectrum of the signal.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Remember $\Omega = \frac{2\pi f}{F_s} = 2\pi F$

To find the inverse relationship, or inverse DTFT, we change the dummy variable *n* to *m* in the above equation and multiply both sides by $e^{j\Omega n}/2\pi$, and then integrate over the interval $-\pi \leq \Omega < \pi$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

$$x[n] = \lim_{\Delta\Omega \to 0} \sum_{k\Delta\Omega=-\pi}^{\pi} \left(X(k\Delta\Omega)\Delta\mathcal{F}\right) e^{j(k\Delta\Omega)n}$$

This *synthesis equation* of the DTFT represents a signal $x[n]$ as a continuous sum (integral) of complex exponentials, each of which is weighted by the signal spectrum $X(\Omega)$.

Example (DTFT)

Determine the DTFT $X(\Omega)$ of a discrete-time rectangular pulse with odd length L_x,

$$x[n] = u[n + (L_x - 1)/2] - u[n - (L_x + 1)/2].$$

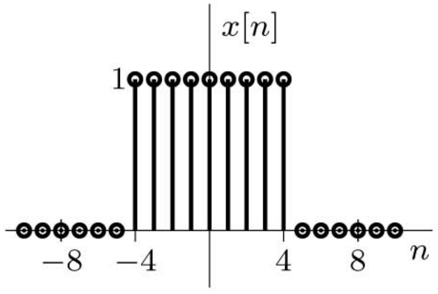
Sketch $x[n]$ and $X(\Omega)$ for $L_x = 9$.

$$\sum_{m=p}^{n} r^m = \frac{r^p - r^{n+1}}{1-r}$$

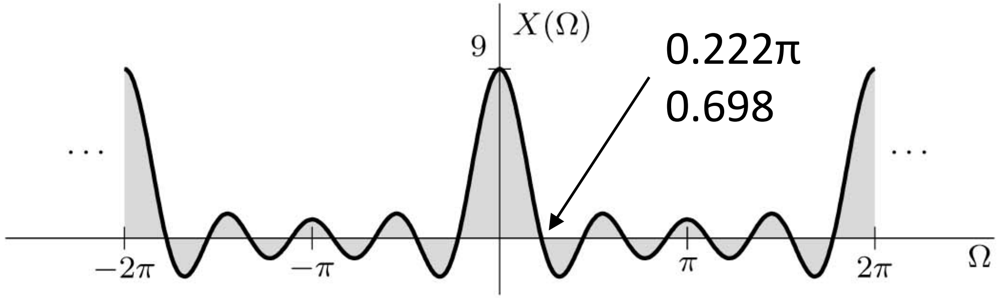
Solution

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$X(\Omega) = \frac{\sin(L_x \Omega/2)}{\sin(\Omega/2)}$$



(a)



(b)

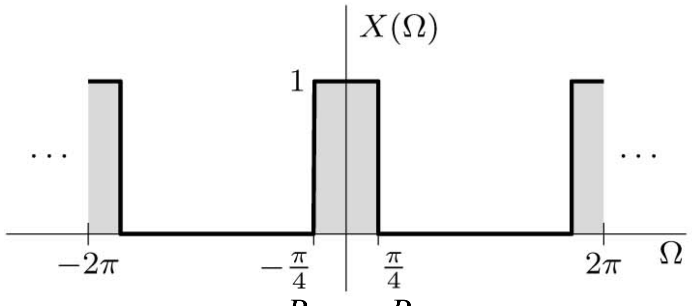
Example (IDTFT)

Determine the IDTFT $x[n]$ of the 2π-periodic rectangular pulse train described over the fundamental band $|\Omega| \leq \pi$ by $X(\Omega) = \Pi(\,\Omega/2B\,)$, where $B \leq \pi$. Sketch $X(\Omega)$ and $x[n]$ for $B = \pi/4$.

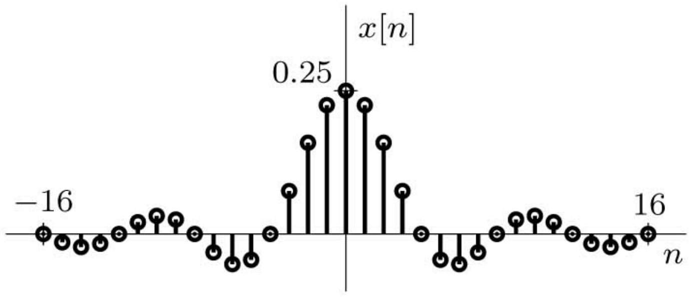
Solution

Note: $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

$$x[n] = \frac{B}{\pi}\,\text{sinc}\left(\frac{Bn}{\pi}\right)$$



(a)



(b)

Magnitude, Phase, and Existence of the DTFT

Magnitude and Phase Spectrum of $X(\Omega)$

$$X(\Omega) = |X(\Omega)|e^{j\angle X(\Omega)}$$

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \angle X(\Omega) = -\angle X(-\Omega)$$

Even magnitude

Odd Phase

Existence

The DTFT exist if $\displaystyle\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

$x[n] = 0.8^n u[n]$

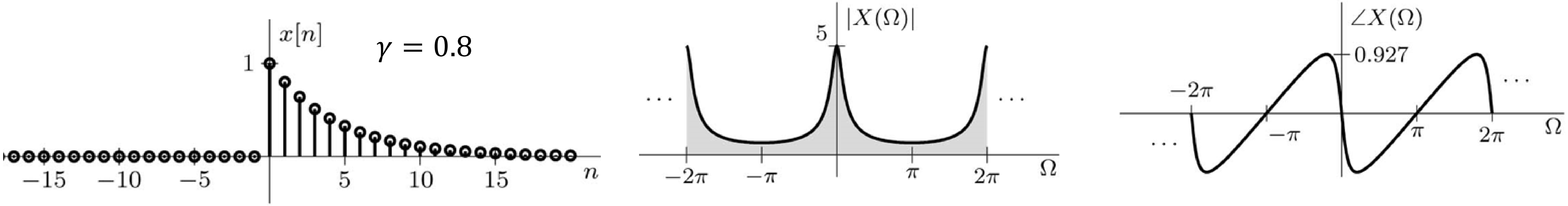
$g[n] = 2^n u[n].$

Example (Spectrum and Phase)

Ex1: Determine the magnitude and phase spectra of the causal exponential $x[n] = \gamma^n u[n]$.

Solution
$$X(\Omega) = \frac{1}{1-\gamma e^{-j\Omega}} \qquad \text{For } |\gamma| < 1$$

$$|X(\Omega)| = \frac{1}{\sqrt{1+\gamma^2 - 2\gamma\cos(\Omega)}} \quad \text{and} \quad \angle X(\Omega) = -\tan^{-1}\left(\frac{\gamma\sin(\Omega)}{1-\gamma\cos(\Omega)}\right)$$



Obtaining the DTFT from CTFT

Synthesis CTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Synthesis DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

If the spectrum of $X(\omega)$ is bandlimited to $|\omega| < \infty$ naturally or by frequency scaling, then the DTFT can be obtained from the CTFT by replacing t by n and ω by Ω.

Example

For each of the following signals, determine the DTFT from the CTFT.

(a) $x[n] = B/\pi \text{ sinc}(Bn/\pi)$

(b) $x[n] = 1$

(c) $x[n] = e^{j\Omega_0 n}$

(d) $x[n] = \cos(\Omega_0 n)$

$$\frac{B}{\pi} \text{sinc}\left(\frac{Bt}{\pi}\right) \overset{FT}{\leftrightarrow} \Pi\left(\frac{\omega}{2B}\right)$$

Read example in textbook

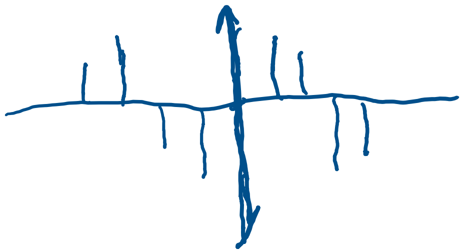
	$x[n]$	$X(\Omega)$	
1.	$\delta[n-k]$	$e^{-jk\Omega}$	Integer k
2.	$\gamma^n u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\|\gamma\| < 1$
3.	$-\gamma^n u[-n-1]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\|\gamma\| > 1$
4.	$\gamma^{\|n\|}$	$\frac{1-\gamma^2}{1-2\gamma\cos(\Omega)+\gamma^2}$	$\|\gamma\| < 1$
5.	$n\gamma^n u[n]$	$\frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$	$\|\gamma\| < 1$
6.	$\|\gamma\|^n \cos(\Omega_0 n+\theta)u[n]$	$\frac{e^{j\Omega}[e^{j\Omega}\cos(\theta)-\|\gamma\|\cos(\Omega_0-\theta)]}{e^{j2\Omega}-2\|\gamma\|\cos(\Omega_0)e^{j\Omega}+\|\gamma\|^2}$	$\|\gamma\| < 1$
7.	$u[n]-u[n-L_x]$	$\frac{\sin(L_x\Omega/2)}{\sin(\Omega/2)}\, e^{-j\Omega(L_x-1)/2}$	
8.	$\frac{B}{\pi}\,\text{sinc}\left(\frac{Bn}{\pi}\right)$	$\sum_{k=-\infty}^{\infty} \Pi\left(\frac{\Omega-2\pi k}{2B}\right)$	
9.	$\frac{B}{2\pi}\,\text{sinc}^2\left(\frac{Bn}{2\pi}\right)$	$\sum_{k=-\infty}^{\infty} \Lambda\left(\frac{\Omega-2\pi k}{2B}\right)$	
10.	1	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
11.	$u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
12.	$e^{j\Omega_0 n}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)$	
13.	$\cos(\Omega_0 n)$	$\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)$	
14.	$\sin(\Omega_0 n)$	$\frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)-\delta(\Omega+\Omega_0-2\pi k)$	
15.	$\cos(\Omega_0 n)\,u[n]$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1} + \frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)$	
16.	$\sin(\Omega_0 n)\,u[n]$	$\frac{e^{j\Omega}\sin(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1} + \frac{\pi}{2j}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)-\delta(\Omega+\Omega_0-2\pi k)$	

Expressing X(Ω) over the fundamental band to simplify expression by removing the summation symbol that indicated periodicity.

	$x[n]$	$X(\Omega)$ for $-\pi \le \Omega < \pi$	
1.	$\delta[n-k]$	$e^{-jk\Omega}$	Integer k
2.	$\gamma^n u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\lvert\gamma\rvert < 1$
3.	$-\gamma^n u[-n-1]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\lvert\gamma\rvert > 1$
4.	$\gamma^{\lvert n\rvert}$	$\frac{1-\gamma^2}{1-2\gamma\cos(\Omega)+\gamma^2}$	$\lvert\gamma\rvert < 1$
5.	$n\gamma^n u[n]$	$\frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$	$\lvert\gamma\rvert < 1$
6.	$\lvert\gamma\rvert^n \cos(\Omega_0 n+\theta)u[n]$	$\frac{e^{j\Omega}[e^{j\Omega}\cos(\theta)-\lvert\gamma\rvert\cos(\Omega_0-\theta)]}{e^{j2\Omega}-2\lvert\gamma\rvert\cos(\Omega_0)e^{j\Omega}+\lvert\gamma\rvert^2}$	$\lvert\gamma\rvert < 1$
7.	$u[n]-u[n-L_x]$	$\frac{\sin(L_x\Omega/2)}{\sin(\Omega/2)}\, e^{-j\Omega(L_x-1)/2}$	
8.	$\frac{B}{\pi}\,\mathrm{sinc}\left(\frac{Bn}{\pi}\right)$	$\Pi\left(\frac{\Omega}{2B}\right)$	$0 < B \le \pi$
9.	$\frac{B}{2\pi}\,\mathrm{sinc}^2\left(\frac{Bn}{2\pi}\right)$	$\Lambda\left(\frac{\Omega}{2B}\right)$	$0 < B \le \pi$
10.	1	$2\pi\delta(\Omega)$	
11.	$u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\delta(\Omega)$	
12.	$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0)$	$\lvert\Omega_0\rvert < \pi$
13.	$\cos(\Omega_0 n)$	$\pi\left[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\right]$	$\lvert\Omega_0\rvert < \pi$
14.	$\sin(\Omega_0 n)$	$\frac{\pi}{j}\left[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)\right]$	$\lvert\Omega_0\rvert < \pi$
15.	$\cos(\Omega_0 n)\,u[n]$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1} + \frac{\pi}{2}\left[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\right]$	$\lvert\Omega_0\rvert < \pi$
16.	$\sin(\Omega_0 n)\,u[n]$	$\frac{e^{j\Omega}\sin(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1} + \frac{\pi}{2j}\left[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)\right]$	$\lvert\Omega_0\rvert < \pi$

Example (DTFT of finite Duration Signals)

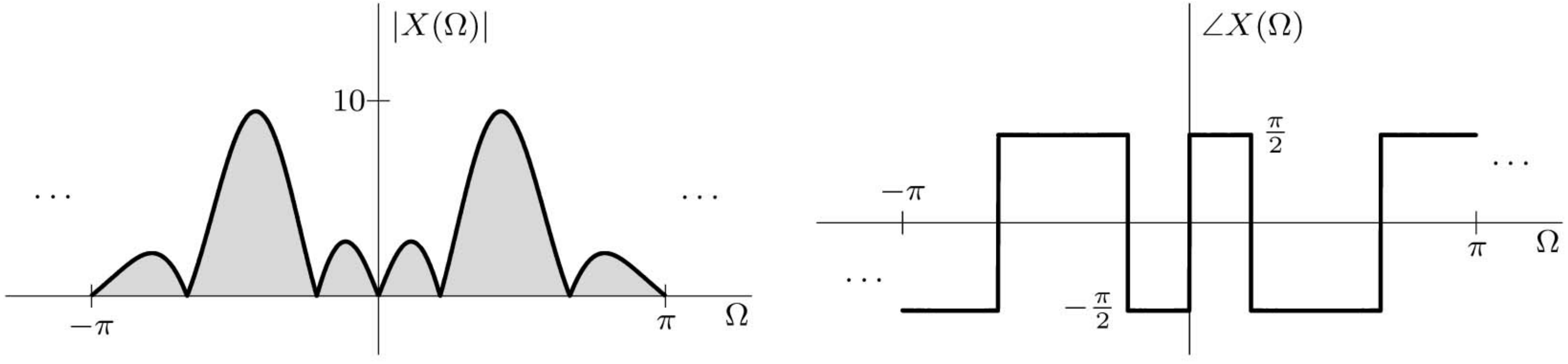
A signal $x[n]$ equals $[1, 2, -1, -2, 0, 2, 1, -2, -1]$ for $-4 \le n \le 4$ and is otherwise 0. Sketch the DTFT of this signal.



$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Solution

```
x = [1 2 -1 -2 0 2 1 -2 -1]; n2 = 4; Omega = linspace(-pi,pi,501);
X = @(Omega) polyval(x,exp(1j*Omega))./exp(1j*Omega*n2);
subplot(211); plot(Omega,abs(X(Omega))); subplot(212);
plot(Omega,angle(X(Omega)));
```



(a) (b)

6.2
Properties of the DTFT

Properties of the DTFT

Linearity Property

If $x[n] \Longleftrightarrow X(\Omega)$ and $y[n] \Longleftrightarrow Y(\Omega)$ then $ax[n] + by[n] \Longleftrightarrow aX(\Omega) + bY(\Omega)$

Complex-Conjugation Property

if $x[n] \Longleftrightarrow X(\Omega)$, then $x^*[n] \Longleftrightarrow X^*(-\Omega)$

Time Scaling Property

No simple rule for time scaling (upsampling or downsampling).

Time-Reversal Property

if $x[n] \Longleftrightarrow X(\Omega)$, then $x[-n] \Longleftrightarrow X(-\Omega)$

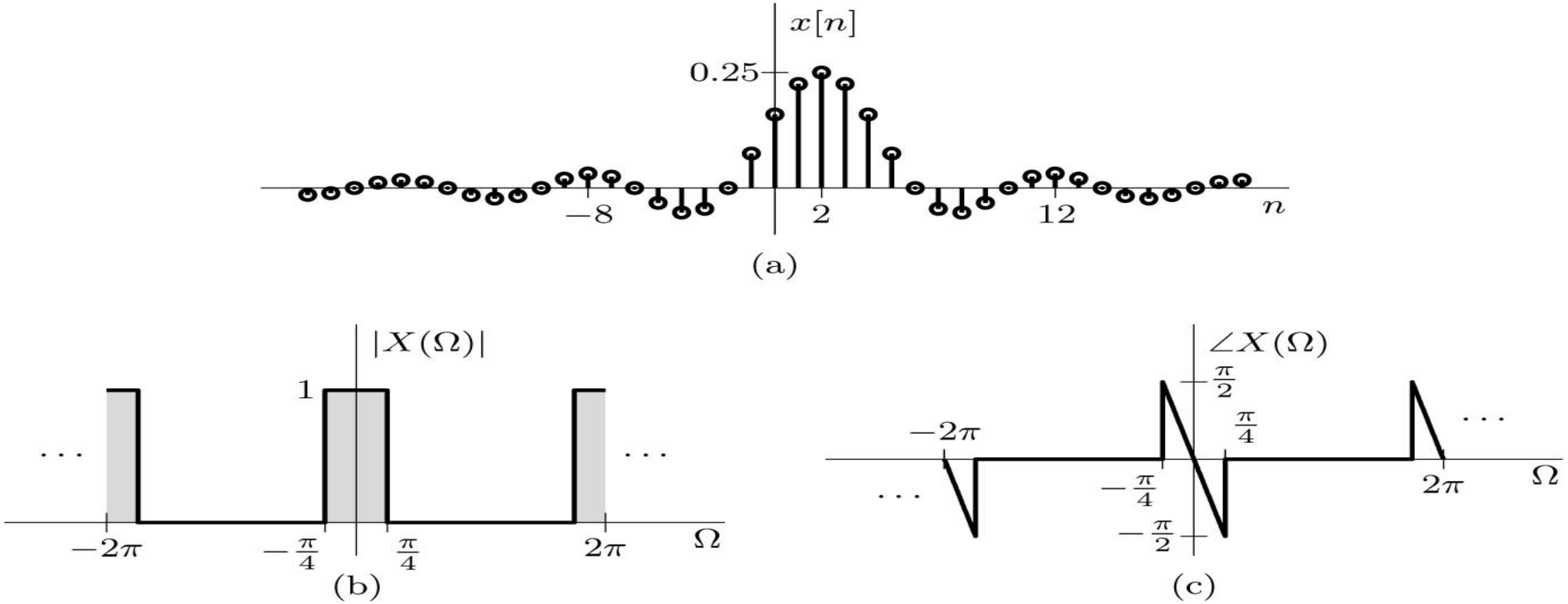
Example: Derive pair 4, $\gamma^{|n|}$, using pair 2, $\gamma^n u(n)$, and the time-reversal property

Properties of the DTFT

Time-Shifting Property

$$\text{if } x[n] \Longleftrightarrow X(\Omega), \text{ then } x[n-m] \Longleftrightarrow X(\Omega)e^{-j\Omega m} \text{ for integer } m$$

Example: Use the time-shifting property to determine the DTFT of $x[n] = \frac{1}{4}\text{sinc}\left(\frac{n-2}{4}\right)$



8. $\frac{B}{\pi}\,\text{sinc}\left(\frac{Bn}{\pi}\right)$ $\Pi\left(\frac{\Omega}{2B}\right)$

Properties of the DTFT

Frequency-Shifting Property

$$\text{if } x[n] \Longleftrightarrow X(\Omega), \text{ then } x[n]e^{j\Omega_0 n} \Longleftrightarrow X(\Omega - \Omega_0)$$

Modulation Property

$$x[n]\cos(\Omega_0 n) \Longleftrightarrow \frac{1}{2}\left[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)\right]$$

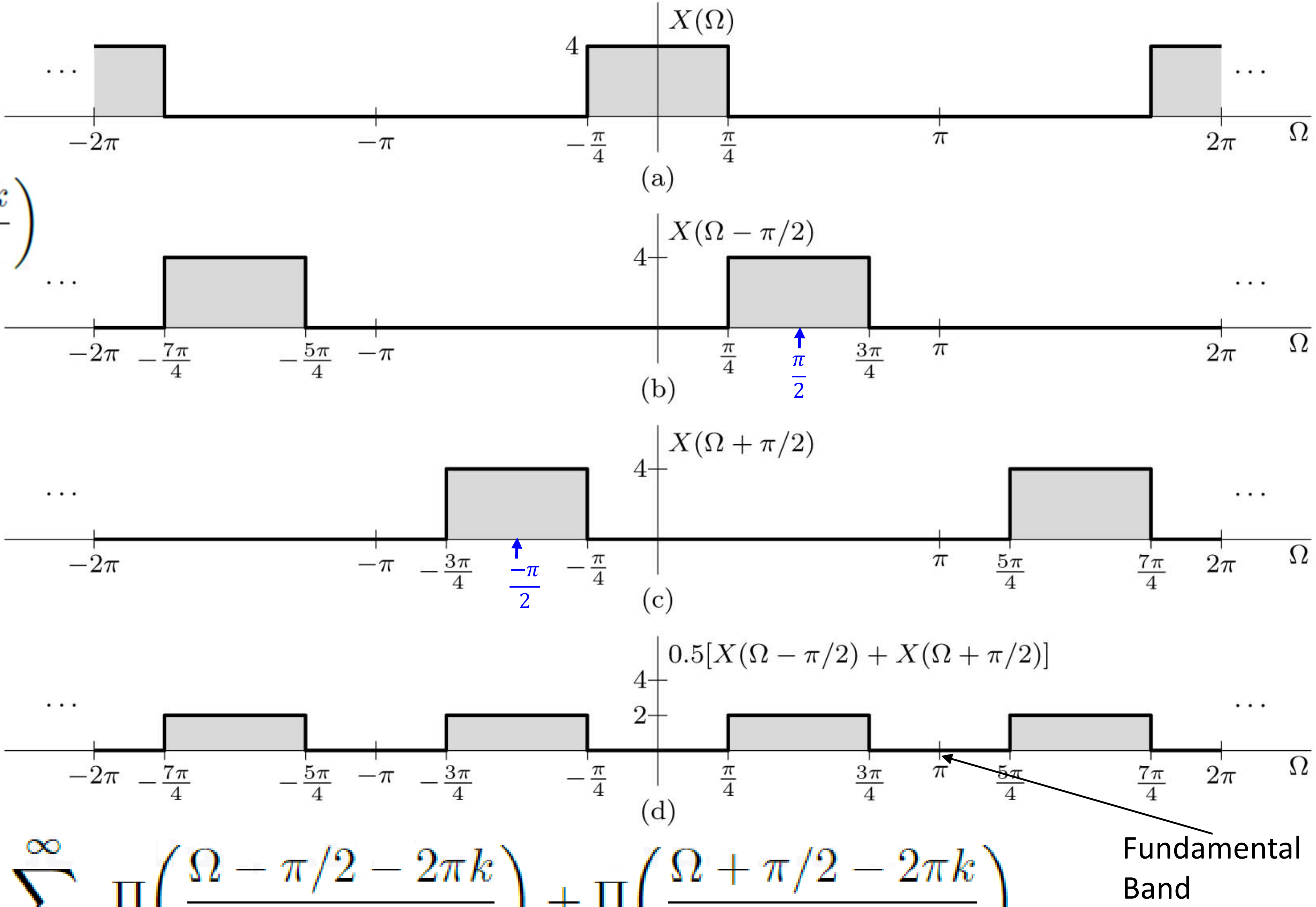
Example: A signal $x[n] = \text{sinc}(n/4)$ modulates a carrier $\cos(\Omega_o n)$. Using the periodic expression for $X(\Omega)$ from the Table, find and sketch the spectrum of the modulated signal $x[n]\cos(\Omega_o n)$ for (a) $\Omega_o = 0.5\pi$ (b) $\Omega_o = 0.85\pi$

8. $\frac{B}{\pi}\text{sinc}\left(\frac{Bn}{\pi}\right)$ $\Pi\left(\frac{\Omega}{2B}\right)$

$x[n] = \mathrm{sinc}(n/4)$

$$X(\Omega) = 4 \sum_{k=-\infty}^{\infty} \Pi\left(\frac{\Omega - 2\pi k}{0.5\pi}\right)$$

$\Omega_o = 0.5\pi$

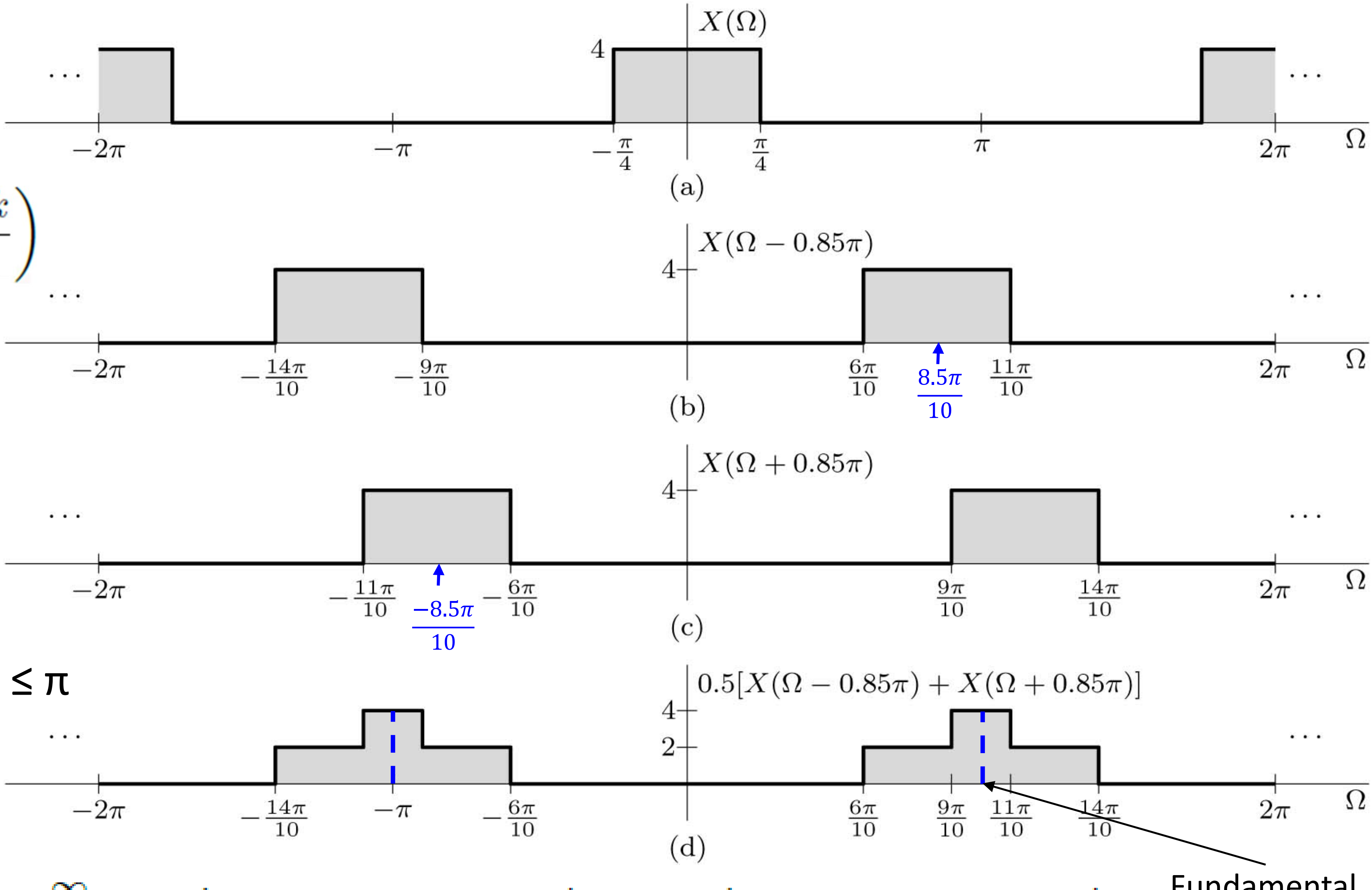


$$x[n]\cos(\pi n/2) \Longleftrightarrow 2 \sum_{k=-\infty}^{\infty} \Pi\left(\frac{\Omega - \pi/2 - 2\pi k}{0.5\pi}\right) + \Pi\left(\frac{\Omega + \pi/2 - 2\pi k}{0.5\pi}\right)$$

$\Omega_o = 0.85\pi$

$$X(\Omega) = 4\sum_{k=-\infty}^{\infty} \Pi\left(\frac{\Omega - 2\pi k}{0.5\pi}\right)$$

To avoid aliasing $(\Omega_0 + B) \leq \pi$



$$x[n]\cos(0.85\pi n) \Longleftrightarrow 2\sum_{k=-\infty}^{\infty} \Pi\left(\frac{\Omega - 0.85\pi - 2\pi k}{0.5\pi}\right) + \Pi\left(\frac{\Omega + 0.85\pi - 2\pi k}{0.5\pi}\right)$$

Properties of the DTFT

Frequency-Differentiation Property

$$\text{if } x[n] \Longleftrightarrow X(\Omega), \text{ then } nx[n] \Longleftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

Example: In the Table, derive pair 5 using pair 2 and the frequency-differentiation property.

pair 2

$$\gamma^n u[n] \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-\gamma}, \qquad |\gamma| < 1$$

pair 5

$$n\gamma^n u[n] \Longleftrightarrow \frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$$

Properties of the DTFT

Time-Domain Convolution Property

$$x[n] \Longleftrightarrow X(\Omega) \quad \text{and} \quad y[n] \Longleftrightarrow Y(\Omega)$$

$$x[n] * y[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m] \Longleftrightarrow X(\Omega)Y(\Omega)$$

This property is the basis for the frequency-domain analysis of LTID systems.

Frequency-Domain Convolution Properties

$$x[n]y[n] \Longleftrightarrow \frac{1}{2\pi} X(\Omega) \circledast Y(\Omega) = \frac{1}{2\pi} \int_{2\pi} X(\lambda)Y(\Omega - \lambda)\, d\lambda$$

Circular (or periodic) convolution

Example

Letting $x[n] = B/\pi \operatorname{sinc}(Bn/\pi)$, find and sketch the spectrum $Y(\Omega)$ of the signal $y[n] = x^2[n]$ for

(a) $0 < B \le \pi/2$ (b) $\pi/2 < B \le \pi$

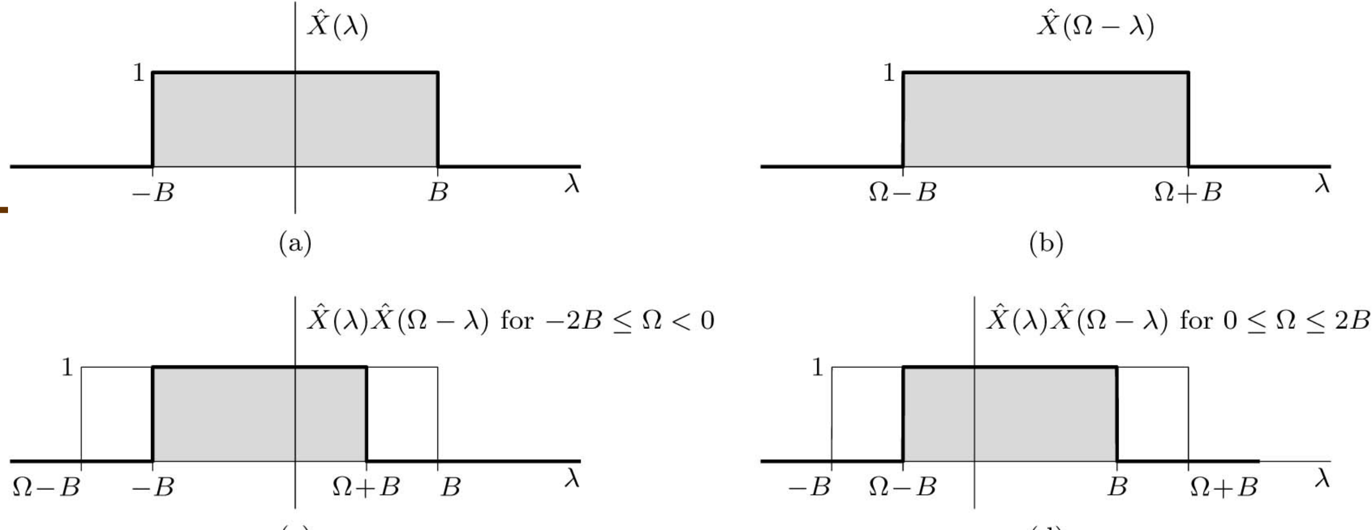
Solution

$$\frac{B}{\pi}\operatorname{sinc}\left(\frac{Bn}{\pi}\right) \Longleftrightarrow \Pi\left(\frac{\Omega}{2B}\right), \qquad |\Omega| \le \pi$$

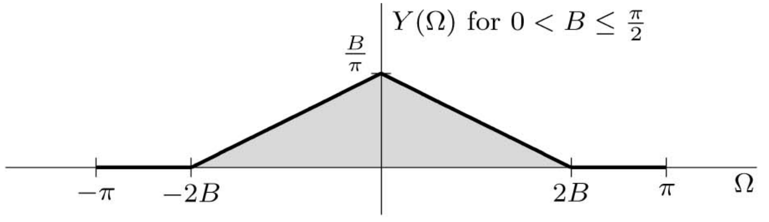
$$\hat{X}(\Omega) * \hat{X}(\Omega) = \int_{-\pi}^{\pi} \hat{X}(\lambda)\hat{X}(\Omega-\lambda)\,d\lambda$$

$$x^2[n] \Longleftrightarrow \frac{1}{2\pi}\hat{X}(\Omega) * \hat{X}(\Omega) = \frac{B}{\pi}\Lambda\left(\frac{\Omega}{4B}\right), \qquad |\Omega| \le \pi$$

$$Y(\Omega) = \frac{B}{\pi}\sum_{k=-\infty}^{\infty}\Lambda\left(\frac{\Omega-2\pi k}{4B}\right)$$



(a) (b) (c) (d)



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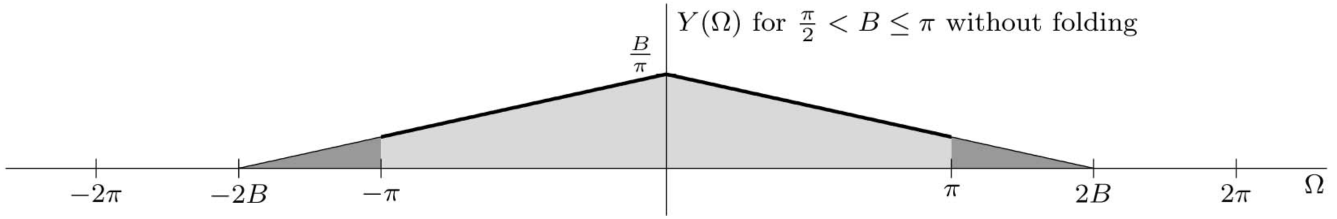

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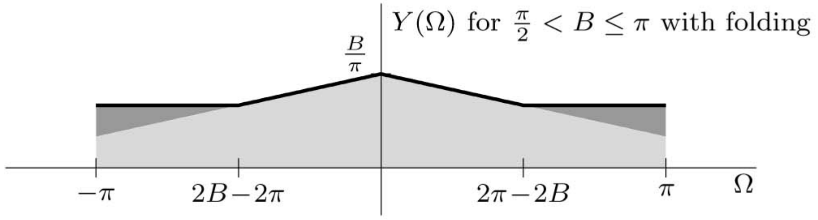


(g)

Properties of the DTFT

Correlation Property

8. $\frac{B}{\pi}\,\mathrm{sinc}\left(\frac{Bn}{\pi}\right)$ $\qquad$ $\Pi\left(\frac{\Omega}{2B}\right)$

$$\rho_{x,y}[l] = \sum_{n=-\infty}^{\infty} x[n+l]y^*[n]$$

$$\rho_{x,y}[l] = x[l] * y^*[-l] \Longleftrightarrow X(\Omega)Y^*(\Omega)$$

Finding Signal Energy in the Frequency-Domain

$$\rho_{x,x}[0] = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \sum_{n=-\infty}^{\infty} |x[n]|^2 = E_x$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2\, d\Omega = \frac{1}{\pi}\int_{0}^{\pi} |X(\Omega)|^2\, d\Omega$$

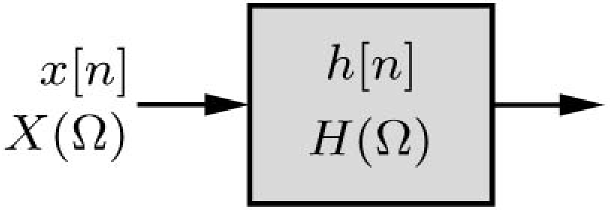
Parseval's Theorem

Example: Use Parseval's theorem to find the energy of $x[n] = \mathrm{sinc}(Bn/\pi)$.

Discrete-Time Fourier Transform	Fourier Transform
Synthesis: $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}\, d\Omega$	**Synthesis:** $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$
Analysis: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$	**Analysis:** $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$
Duality:	**Duality:** if $x(t) \Longleftrightarrow X(\omega)$, then $X(t) \Longleftrightarrow 2\pi x(-\omega)$
Linearity: $ax[n] + by[n] \Longleftrightarrow aX(\Omega) + bY(\Omega)$	**Linearity:** $ax(t) + by(t) \Longleftrightarrow aX(\omega) + bY(\omega)$
Complex Conjugation: $x^*[n] \Longleftrightarrow X^*(-\Omega)$	**Complex Conjugation:** $x^*(t) \Longleftrightarrow X^*(-\omega)$
Scaling and Reversal: (see Sec. 6.6) $x[-n] \Longleftrightarrow X(-\Omega)$	**Scaling and Reversal:** $x(at) \Longleftrightarrow \frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$ $x(-t) \Longleftrightarrow X(-\omega)$
Shifting: $x[n-m] \Longleftrightarrow X(\Omega)e^{-j\Omega m}$ $x[n]e^{j\Omega_0 n} \Longleftrightarrow X(\Omega - \Omega_0)$	**Shifting:** $x(t-t_0) \Longleftrightarrow X(\omega)e^{-j\omega t_0}$ $x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega - \omega_0)$
Differentiation: $-jnx[n] \Longleftrightarrow \frac{d}{d\Omega}X(\Omega)$	**Differentiation:** $\frac{d}{dt}x(t) \Longleftrightarrow j\omega X(\omega)$ $-jtx(t) \Longleftrightarrow \frac{d}{d\omega}X(\omega)$
Time Integration:	**Time Integration:** $\int_{-\infty}^{t} x(\tau)d\tau \Longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Convolution: $x[n] * y[n] \Longleftrightarrow X(\Omega)Y(\Omega)$ $x[n]y[n] \Longleftrightarrow \frac{1}{2\pi}X(\Omega)\circledast Y(\Omega)$	**Convolution:** $x(t) * y(t) \Longleftrightarrow X(\omega)Y(\omega)$ $x(t)y(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * Y(\omega)$
Correlation: $\rho_{x,y}[l] = x[l] * y^*[-l] \Longleftrightarrow X(\Omega)Y^*(\Omega)$	**Correlation:** $\rho_{x,y}(\tau) = x(\tau) * y^*(-\tau) \Longleftrightarrow X(\omega)Y^*(\omega)$
Parseval's: $E_x = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega)\rvert^2\, d\Omega$	**Parseval's:** $E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$

6.3
LTID System Analysis by the DTFT

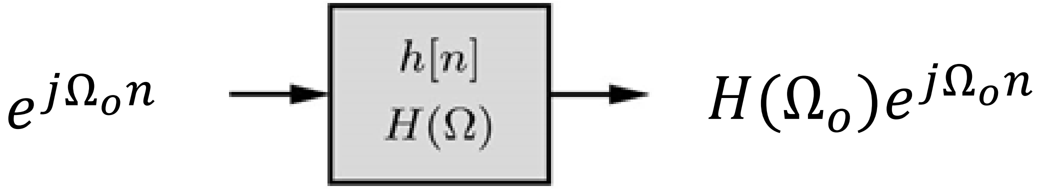
LTID System Analysis



$$y[n] = x[n] * h[n]$$
$$Y(\Omega) = X(\Omega)H(\Omega)$$

$$y[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)H(\Omega)e^{j\Omega n}\, d\Omega$$

The output $y[n]$ is a scaled sum of the responses to all the component frequencies of the input.



If the input is everlasting exponential, $x[n] = e^{j\Omega_o n}$, then the output will be the same as the input multiplied by the system response to the frequency Ω_o, $y[n] = H(\Omega_o)e^{j\Omega_o n}$.

Frequency Response from a Difference Equation

$$\sum_{k=0}^{K} a_k y[n-k] = \sum_{l=0}^{K} b_l x[n-l] \quad \Longleftrightarrow \quad \sum_{k=0}^{K} a_k Y(\Omega) e^{-jk\Omega} = \sum_{l=0}^{K} b_l X(\Omega) e^{-jl\Omega}$$

$$\left(\sum_{k=0}^{K} a_k e^{-jk\Omega}\right) Y(\Omega) = \left(\sum_{l=0}^{K} b_l e^{-jl\Omega}\right) X(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{l=0}^{K} b_l\, e^{-jl\Omega}}{\sum_{k=0}^{K} a_k\, e^{-jk\Omega}}$$

Find $H(\Omega)$ and determine the zero-state response $y[n]$ to input $x[n] = \cos(3\pi n/10)$ of an LTID system described by the difference equation

$y[n] - 1.8y[n-1] + 0.81y[n-2] = x[n] + 2x[n-1]$

Solution

$$y[n] = 3.61 \cos(3\pi n/10 - 2.63)$$

Example

An LTID system is specified by the equation

$$y[n] - 0.5y[n-1] = x[n].$$

Find $H(\Omega)$, the frequency response of this system, and determine the zero-state response $y[n]$ if the input $x[n] = (0.8)^n u[n]$.

Solution

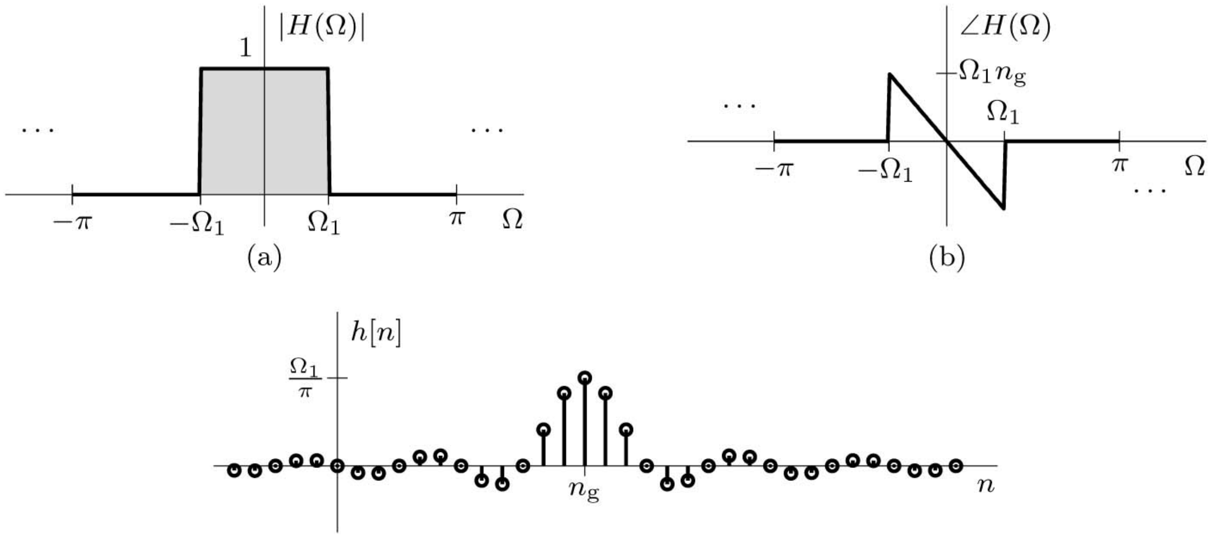
$$y[n] = \left[\frac{8}{3}(0.8)^n - \frac{5}{3}(0.5)^n\right] u[n]$$

$\gamma^n u[n]$ $\qquad$ $\frac{e^{j\Omega}}{e^{j\Omega} - \gamma}$

Matlab Code to find Partial Fraction Expansion

```
[r,p,k] = residuez(1,poly([0.8,0.5]))
r = 2.6667 -1.6667
p = 0.8000 0.5000
k = []
```

Ideal and Realizable Filters

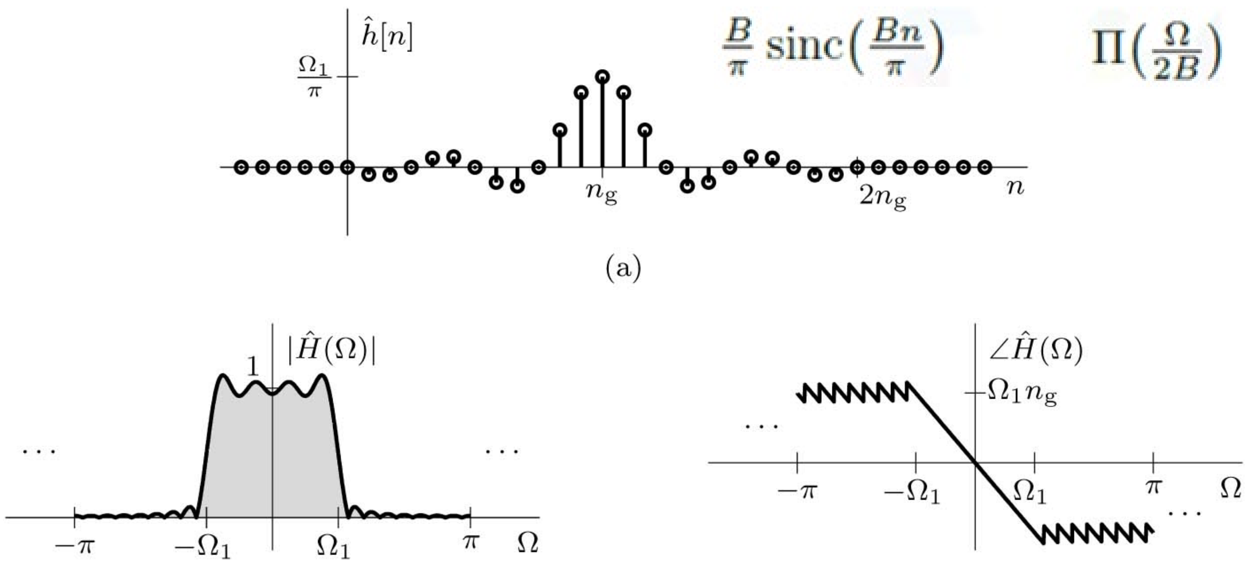


(a) (b) (c)

$$H(\Omega) = \sum_{k=-\infty}^{\infty} \Pi\left(\frac{\Omega - 2\pi k}{2\Omega_1}\right) e^{-j\Omega n_g}$$

$$h[n] = \frac{\Omega_1}{\pi} \operatorname{sinc}\left[\frac{\Omega_1(n - n_g)}{\pi}\right]$$

Ideal Lowpass Filter: Noncausal System



(a) (b) (c)

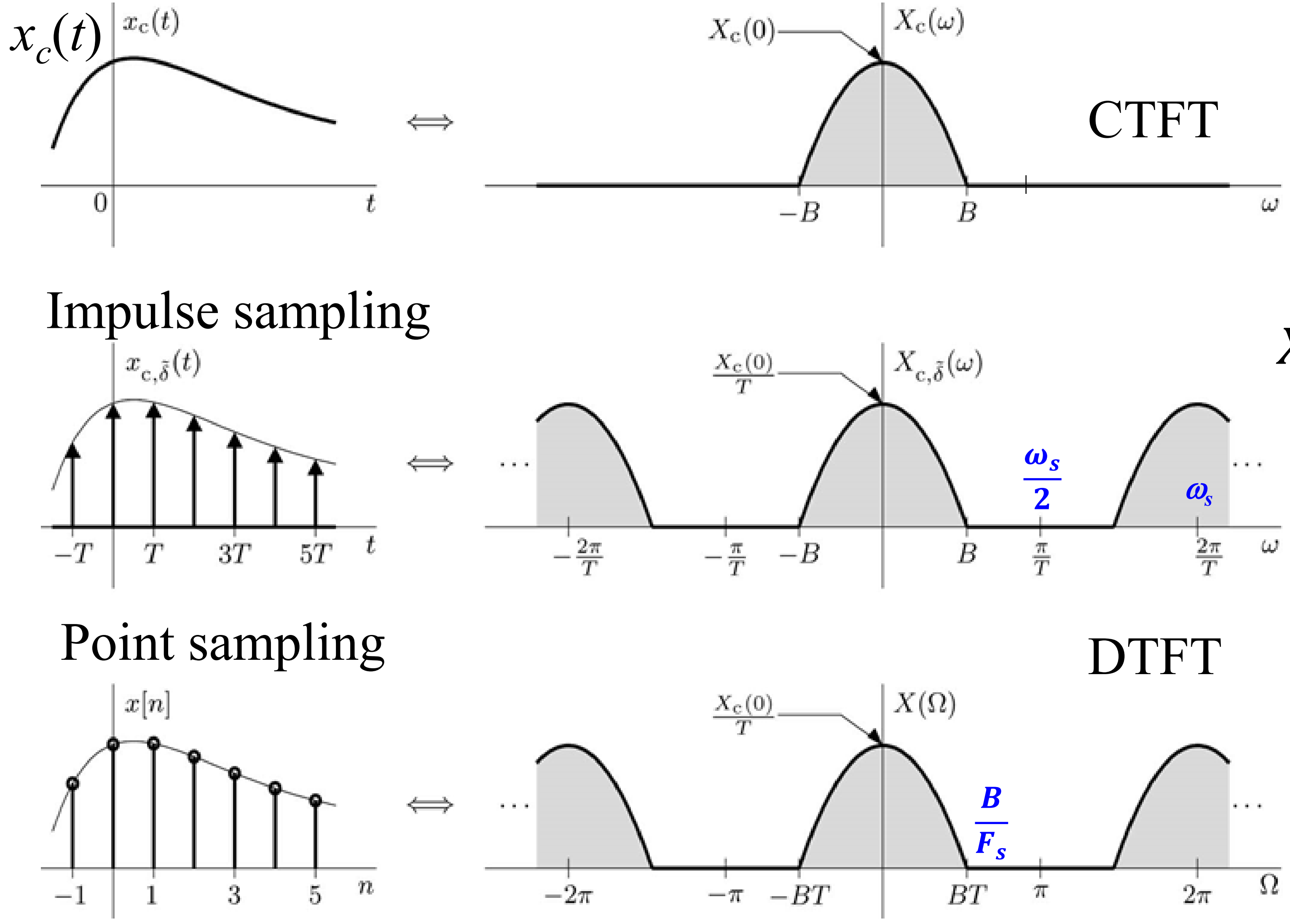
$$\hat{h}[n] = h[n]\,\Pi\left(\frac{n - n_g}{2n_g^{+}}\right)$$

$$\hat{H}(\Omega) = \sum_{n=-\infty}^{\infty} \hat{h}[n] e^{-j\Omega n}$$

Realizable Lowpass Filter: with delay and ripple in pass band region.

6.4
Connection between the DTFT and the CTFT

Connection between the DTFT and the CTFT



$T = 1/f_s \qquad \omega_s = 2\pi/T$

$$X(\omega T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\omega - k\omega_s)$$

$\Omega = \omega T$

Replace ω with Ω/T

$$X(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\Omega - 2\pi k}{T}\right)$$

$$X(\Omega) = \frac{1}{T} X_c\left(\frac{\Omega}{T}\right)$$

The DTFT of a DT signal $x[n]$ equals a scaled periodic replication of the CTFT of the CT signal $x(t)$ sampled at T.

Example

Consider a bandlimited signal $x_c(t)$ with spectrum

$$X_c(\omega) = \begin{cases} \frac{100}{\omega^2+10^4} & |\omega| < 50\pi \\ 0 & \text{otherwise} \end{cases}$$

For $T = 0.01$, determine the DTFT $X(\Omega)$ of $x[n] = x_c(nT)$. Discuss what happens if instead $T = 0.1$.

$$X(\Omega) = \frac{1}{T} X_c\left(\frac{\Omega}{T}\right)$$

$$X(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\Omega - 2\pi k}{T}\right)$$

Answer

$$X(\Omega) = \begin{cases} \frac{1}{\Omega^2 + 1} & |\Omega| < 0.5\pi \\ 0 & \text{otherwise} \end{cases}$$

6.5
Digital Processing of Analog Signals

$x(t)$

$X_c(\omega)$

$h_c(t)$

$H_c(\omega)$

$y(t)$

$Y_c(\omega)$

$\Omega = \omega T$

$x[n] = x(nT)$

$X(\Omega) = \frac{1}{T} X_c\left(\frac{\Omega}{T}\right), \quad |\Omega| \leq \pi$

$h[n] = T h_c(nT)$

$H(\Omega) = \hat{H}_c(\Omega/T), \ |\Omega| \leq \pi$

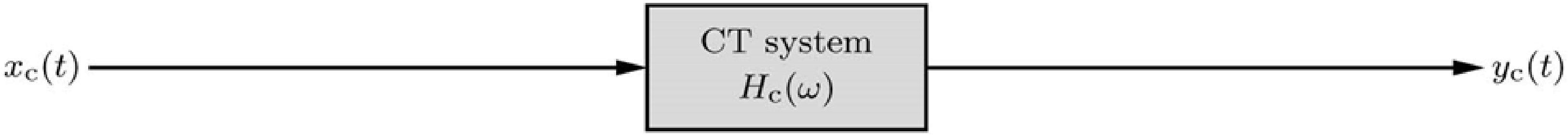
$y[n] = y(nT)$

$Y(\Omega) = \frac{1}{T} Y_c\left(\frac{\Omega}{T}\right) \quad |\Omega| \leq \pi$

Digital Processing of Analog Signals

Important conditions for the impulse invariance method: (design of discrete IIR from continuous)

1. The sampling rate is above the Nyquist rate, higher than twice the bandwidth of the signal $x(t)$
2. The system is bandlimited



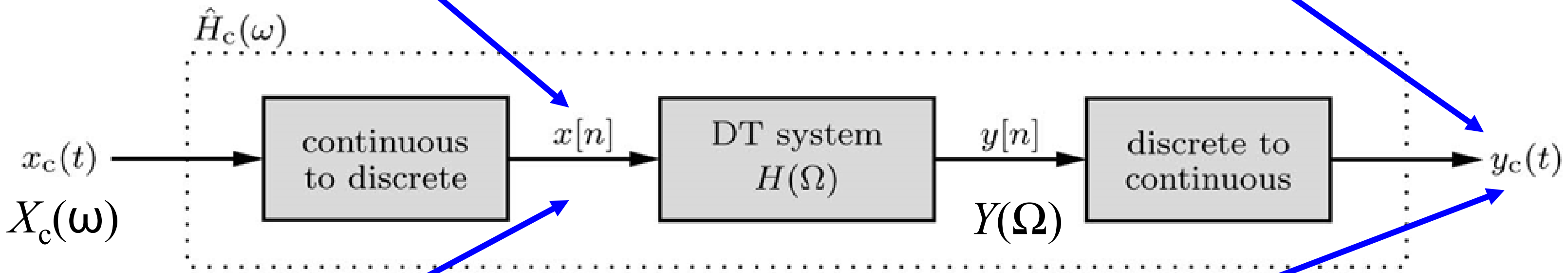
Time domain Characterization of ideal C/D and D/C Converter

$$\hat{H}_c(\omega) = \begin{cases} H_c(\omega) & |\omega| \le \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

$$H(\Omega) = \hat{H}_c(\Omega/T), \quad |\Omega| \le \pi$$

$$x[n] = x_c(nT)$$

$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \, \text{sinc}\left(\frac{t - nT}{T}\right)$$



$$X(\Omega) = \frac{1}{T} X_c\left(\frac{\Omega}{T}\right), \quad |\Omega| \le \pi$$

$$Y_c(\omega) = T Y(\omega T), \quad |\omega T| \le \pi$$

Frequency domain Characterization of ideal C/D and D/C Converter

Digital Filter Design from Continuous Filter $H_c(\omega)$

Frequency-domain criterion for digital filter design:

$$H(\Omega) = H_c(\Omega/T), \qquad |\Omega| \le \pi$$

Prove

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\frac{1}{T} Y_c(\Omega/T)}{\frac{1}{T} X_c(\Omega/T)} = H_c(\Omega/T)$$

Time-domain criterion for digital filter design (IIR impulse invariance method):

$$h[n] = T h_c(nT)$$

Prove

$$h_c(t) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} H_c(\omega) e^{j\omega t}\, d\omega \xrightarrow[\omega \to \Omega/T]{t \to nT} h_c(nT) = \frac{1}{2\pi T} \int_{-\pi}^{\pi} H(\Omega) e^{j\Omega n}\, d\Omega = \frac{1}{T} h[n]$$

Example

Using the minimum possible sampling frequency, design a DT system that delays a bandlimited ($B \leq 10$ kHz) CT input $x_c(t)$ by $\tau = 250\ \mu$s. Sketch the frequency responses $H(\Omega)$ and $\widehat{H}_c(\omega)$. Explain how closely the system compares with an ideal analog delay $\tau = 250\ \mu$s. Using a hypothetical bandlimited ($B \leq 10$ kHz) signal $x_c(t)$ and its spectrum $X_c(\omega)$, sketch the signals and spectra at the input of the C/D converter, the input of $H(\Omega)$, the output of $H(\Omega)$, and the output of the D/C converter.

Answer

$$y_c(t) = x_c(t - \tau)$$

$$H_c(\omega) = e^{-j\omega\tau}$$

$$H(\Omega) = H_c(\Omega/T)$$

$$H(\Omega) = e^{-j\Omega\tau/T} = e^{-j5\Omega} \qquad |\Omega| \leq \pi$$

$$H_c(\omega) = e^{-j\omega\tau} = e^{-j5\omega T}$$

$$H(\Omega) = H_c(\Omega/T) = e^{-j5\Omega}, \qquad |\Omega| \le \pi$$

$$h[n] = \delta[n-5]$$

$$\hat{H}_c(\omega) = \begin{cases} e^{-j5\omega T} & |\omega T| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{H}_c(\omega) = e^{-j\omega/4000}$$

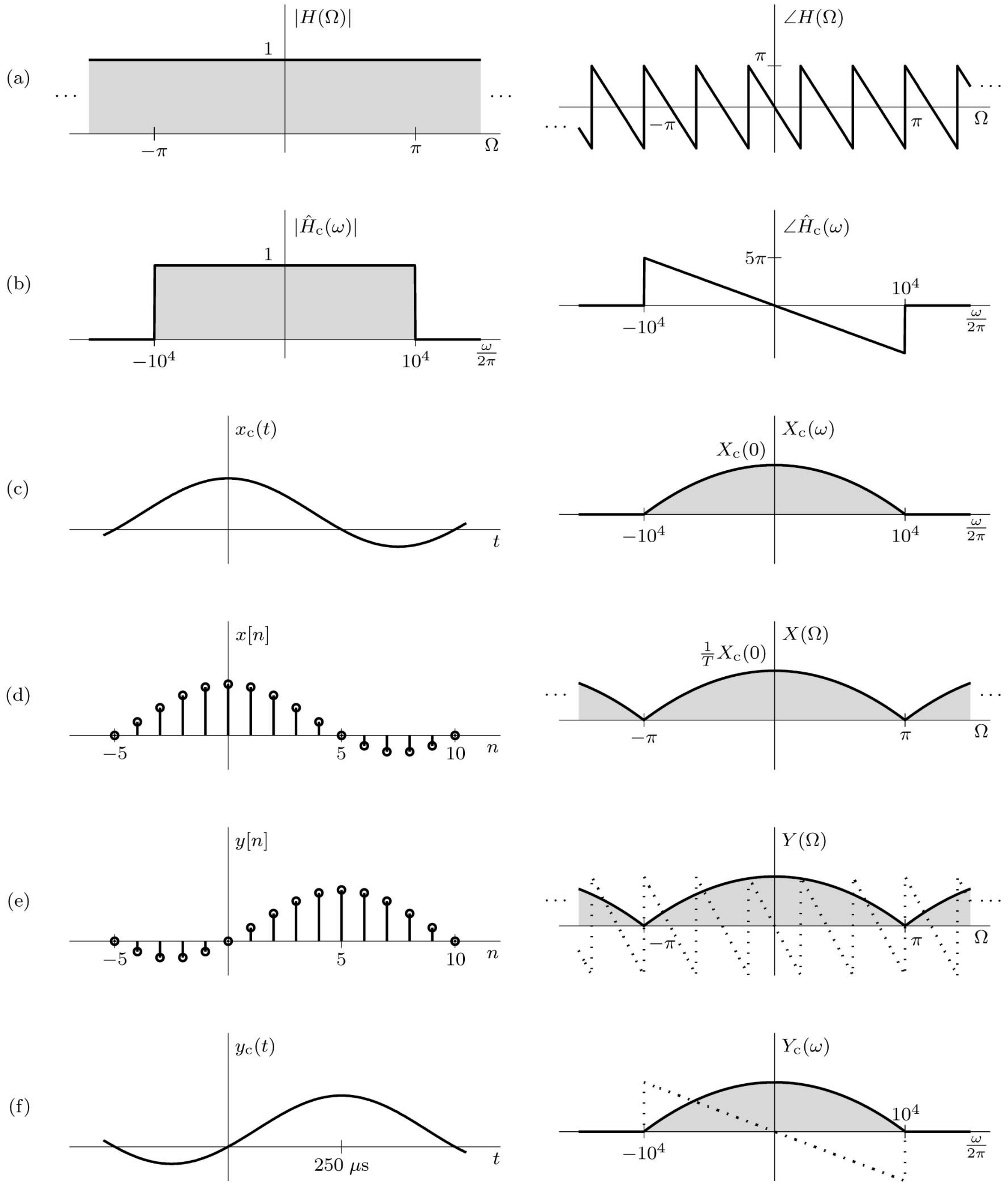
$$X(\Omega) = \frac{1}{T} X_c(\Omega/T), \qquad |\Omega| \le \pi$$

$$Y(\Omega) = X(\Omega)H(\Omega) = X(\Omega)\, e^{-j5\Omega}$$

$$y[n] = x[n-5]$$

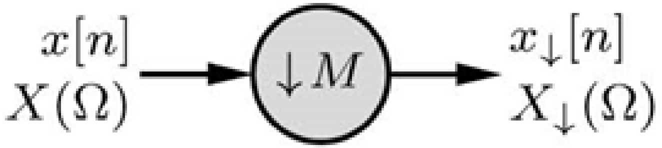
$$Y_c(\omega) = X_c(\omega)\, e^{-j5\omega T}$$

$$y_c(t) = x_c(t-5T) = x_c(t-\tau), \qquad \tau = 250\,\mu\text{s}.$$

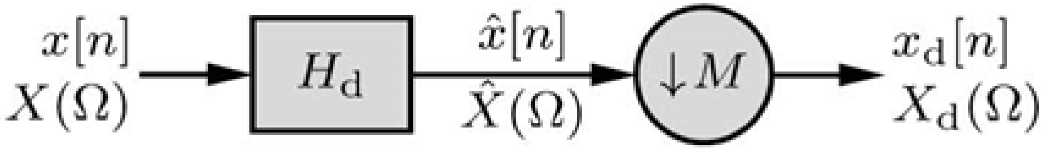


6.6
Digital Resampling:
A Frequency-Domain Perspective

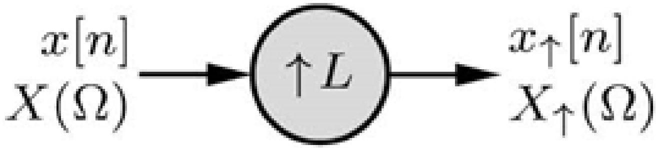
Digital Resampling: A Frequency-Domain Perspective



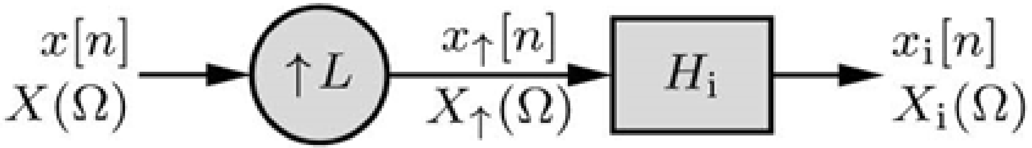
Downsampling



Decimation

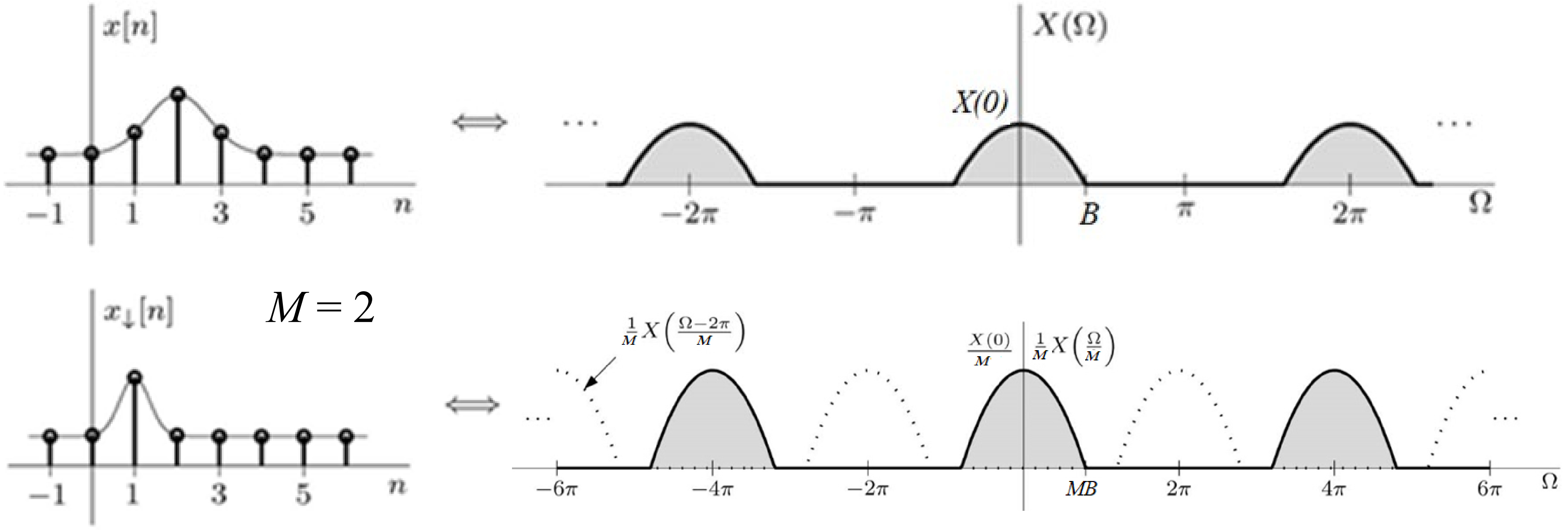


Upsampling



Interpolation

Down-Sampling



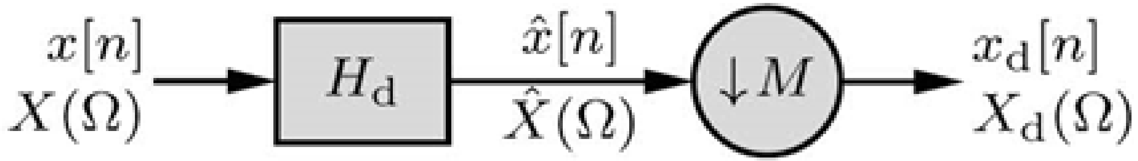
$$x_{\downarrow}[n] = x[Mn]$$

$$X_{\downarrow}(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\Omega - 2\pi m}{M}\right)$$

To avoid aliasing $MB \leq \pi$

Downsampling and Decimation

If the spectrum $X_{\downarrow}(\Omega)$ of the downsampled signal $x[n]$ does not meet this condition $MB \leq \pi$ then use a filter H_d with cutoff frequency $\Omega_c \leq \pi/M$.



$$\hat{X}(\Omega) = X(\Omega)H_d(\Omega)$$

Decimation

$$X_{\mathrm{d}}(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} \hat{X}\left(\frac{\Omega - 2\pi m}{M}\right) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\Omega - 2\pi m}{M}\right) H_{\mathrm{d}}\left(\frac{\Omega - 2\pi m}{M}\right) \qquad \text{for } MB \leq \pi$$

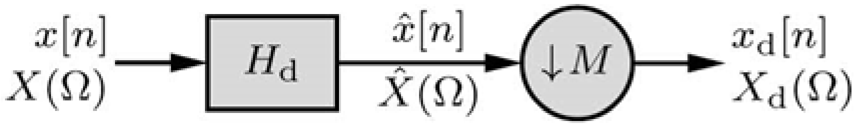
Time representation for Decimation

$$h_{\mathrm{d}}[n] = \frac{1}{M}\mathrm{sinc}(n/M) \qquad x_{\mathrm{d}}[n] = \frac{1}{M}\sum_{k=-\infty}^{\infty} x[k]\,\mathrm{sinc}\left(n - \frac{k}{M}\right)$$

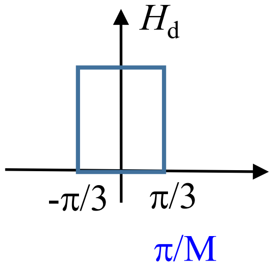
Downsampling and Decimation

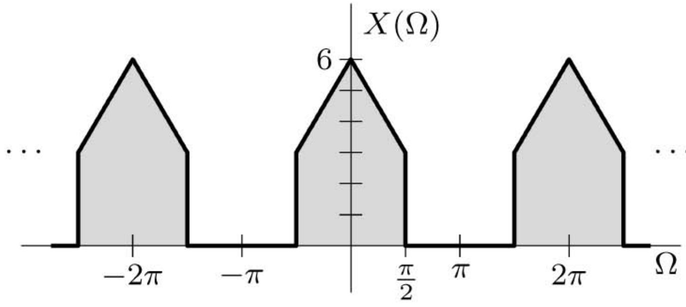
The DTFT $X(\Omega)$ of a signal $x[n]$ is shown below. Using $M = 3$, determine the DTFTs of the downsampled signal $x_{\downarrow}[n]$ and the decimated signal $x_d[n]$. Assume that the decimator uses an ideal lowpass filter.

$$X_{\downarrow}(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\Omega - 2\pi m}{M}\right)$$

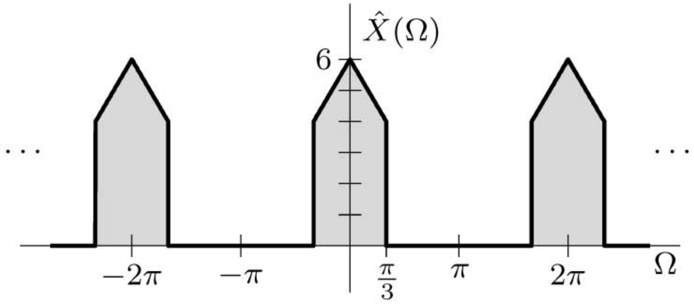


Decimation



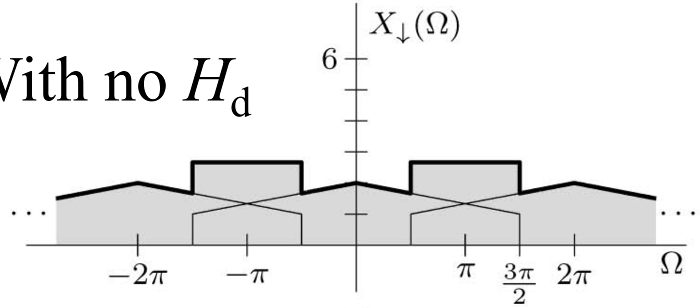


(a)

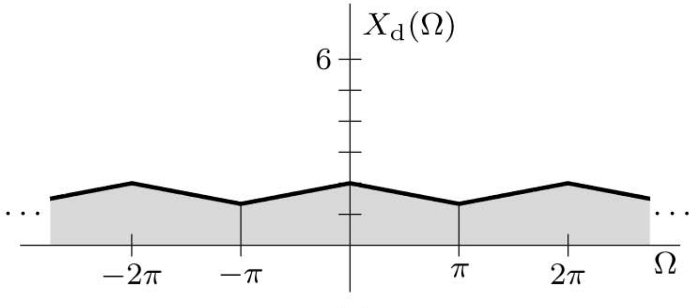


(c)

With no H_d



(b)



(d)

Downsampling and Decimation

A signal $x[n] = \gamma^n u[n]$, where $|\gamma| < 1$, is downsampled by factor $M = 2$. Determine the resulting downsampled signal $x_\downarrow[n] = x[2n]$ and its spectrum $X_\downarrow(\Omega)$.

$$x[n] = \gamma^n u[n] \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} = \frac{1}{1 - \gamma e^{-j\Omega}} = X(\Omega)$$

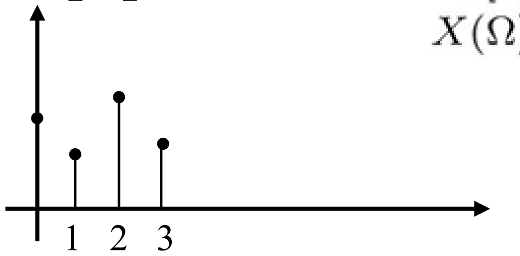
$$x_\downarrow[n] = x[Mn] = x[2n] = \gamma^{2n} u[2n] = \gamma^{2n} u[n]$$

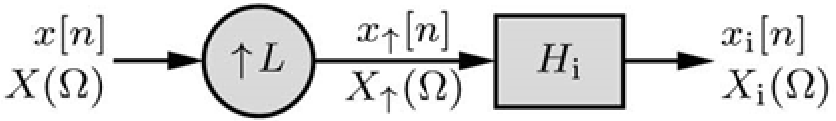
We can use the table to find the $X_\downarrow(\Omega)$ but let us use $X_\downarrow(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\Omega - 2\pi m}{M}\right)$

$$\begin{aligned}
X_\downarrow(\Omega) &= \frac{1}{2}\left[X\left(\frac{\Omega}{2}\right) + X\left(\frac{\Omega - 2\pi}{2}\right)\right] \\
&= \frac{1}{2}\left[X\left(\frac{\Omega}{2}\right) + X\left(\frac{\Omega}{2} - \pi\right)\right] \\
&= \frac{1}{2(1 - \gamma e^{-j\Omega/2})} + \frac{1}{2(1 + \gamma e^{-j\Omega/2})} \\
&= \frac{1}{1 - \gamma^2 e^{-j\Omega}}.
\end{aligned}$$

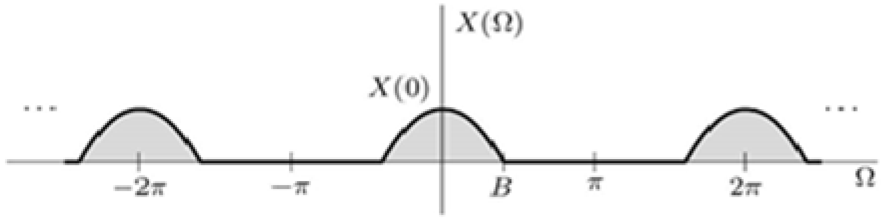
Interpolation and Upsampling

$x[n]$

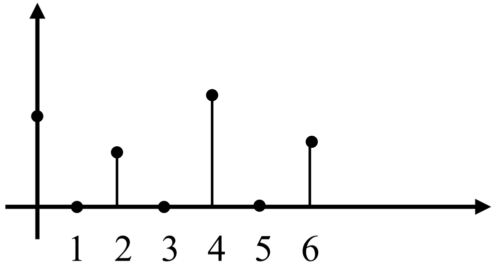




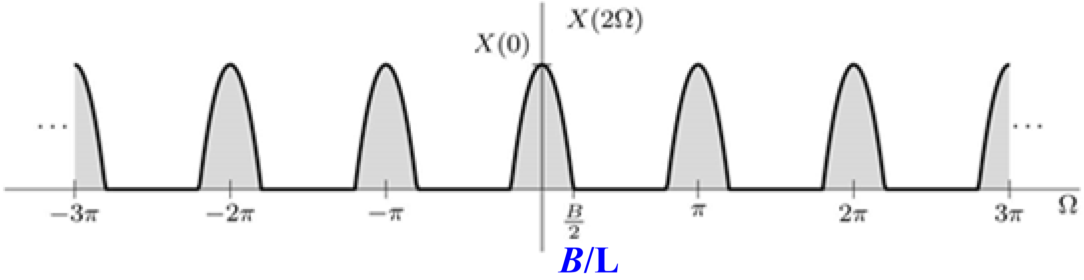
Interpolation



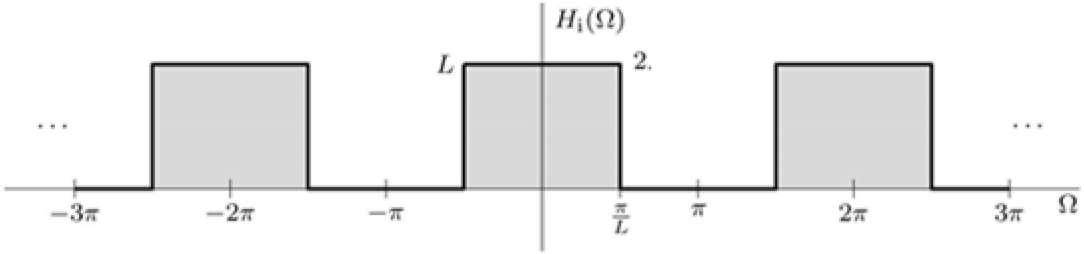
$x_\uparrow[n] = x[n/L]$

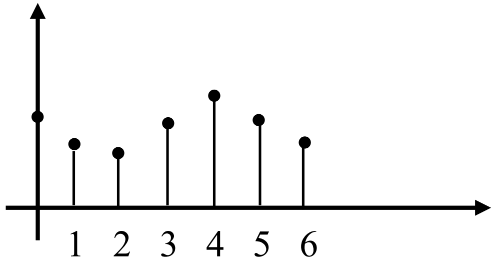


$$X_\uparrow(\Omega) = X(L\Omega)$$

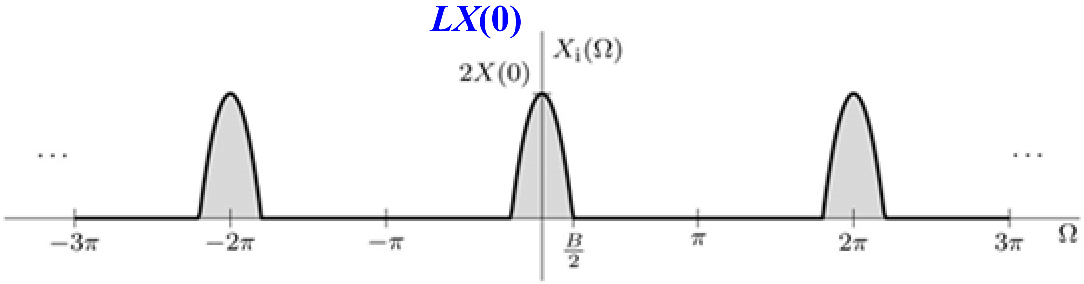


$$H_i(\Omega) = L \sum_{k=-\infty}^{\infty} \Pi\left(\frac{\Omega - 2\pi k}{2\pi/L}\right)$$



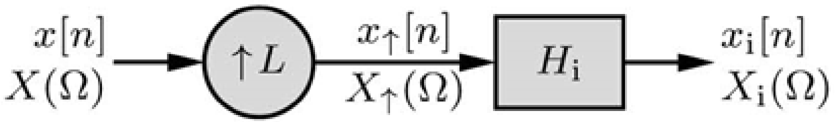


$$X_i(\Omega) = X_\uparrow(\Omega) H_i(\Omega)$$



Interpolation and Upsampling

Consider a signal $x[n] = \gamma^{2n}u[n]$, where $|\gamma| < 1$.
Using $L = 2$, determine the spectra of the upsampled signal $x_{\uparrow}[n]$ and the ideally interpolated signal $x_{i}[n]$.



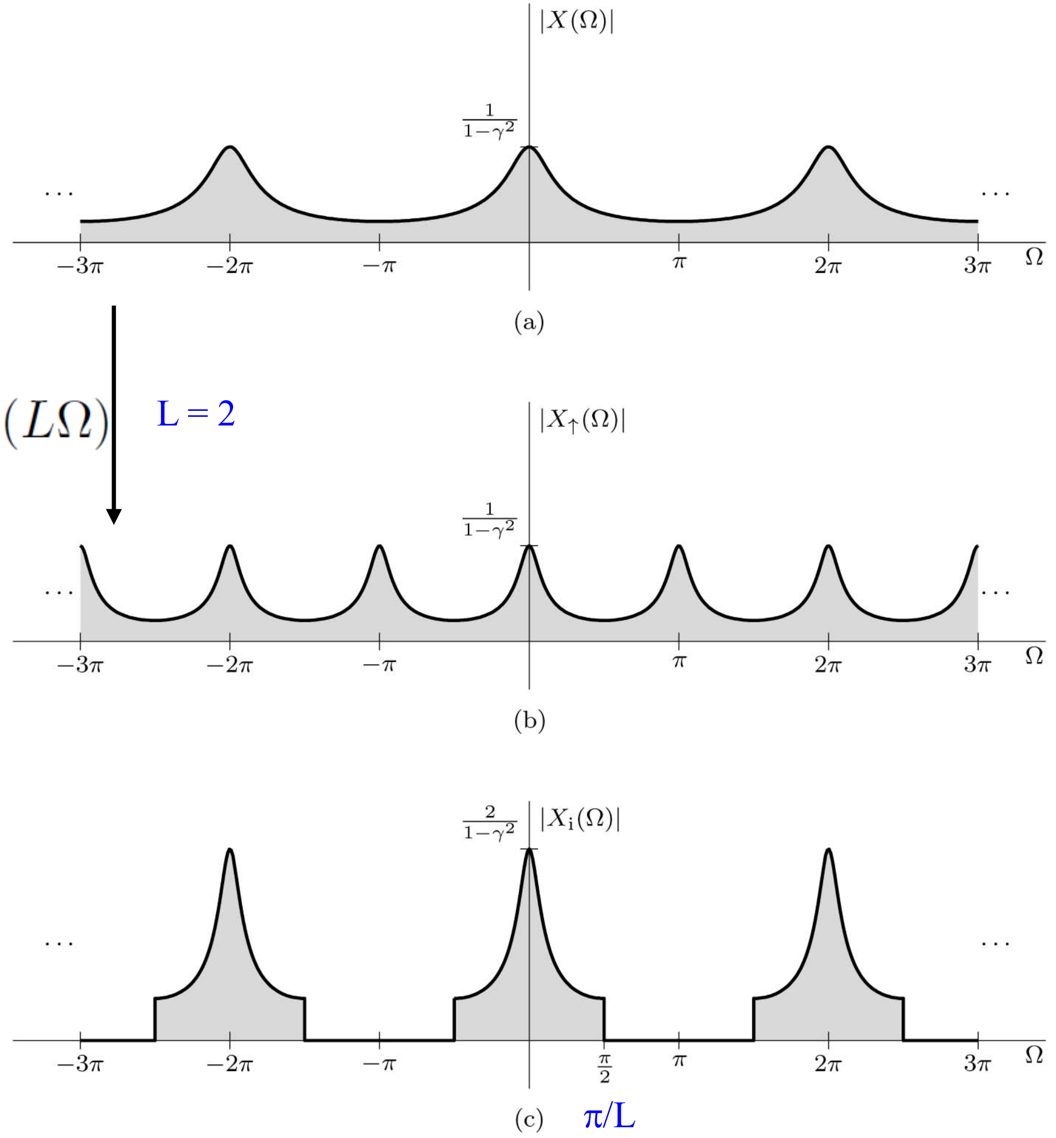
Interpolation

$$X(\Omega) = 1/(1 - \gamma^2 e^{-j\Omega})$$

$$X_{\uparrow}(\Omega) = X(2\Omega) = \frac{1}{1-\gamma^2 e^{-j2\Omega}}$$

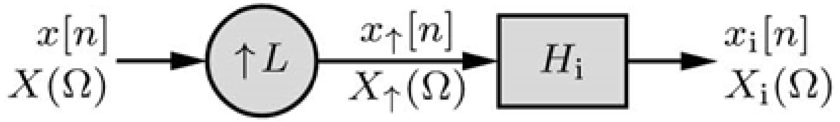
$$X_{i}(\Omega) = X_{\uparrow}(\Omega) H_{i}(\Omega) = \frac{2}{1-\gamma^2 e^{-j2\Omega}} \Pi\left(\frac{\Omega}{\pi}\right), \qquad |\Omega| \leq \pi$$

$$X_{\uparrow}(\Omega) = X(L\Omega)$$

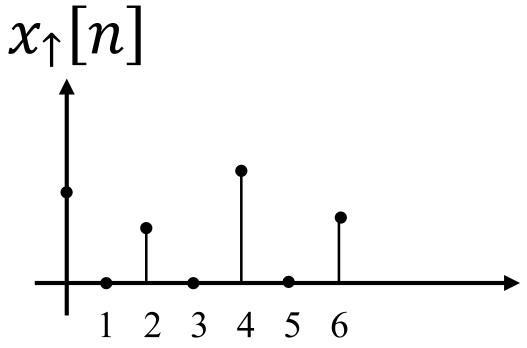


Time-Domain Characterization of Up-Sampling

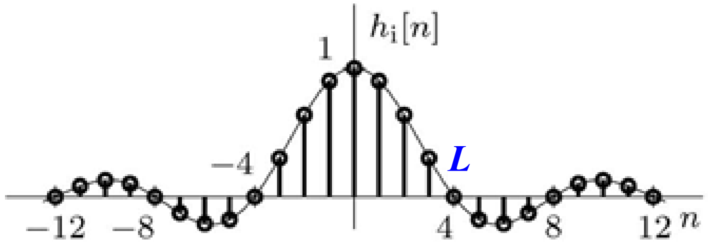
$$x_i[n] = x_\uparrow[n] * h_i[n] = \sum_{m=-\infty}^{\infty} x_\uparrow[m] h_i[n-m]$$



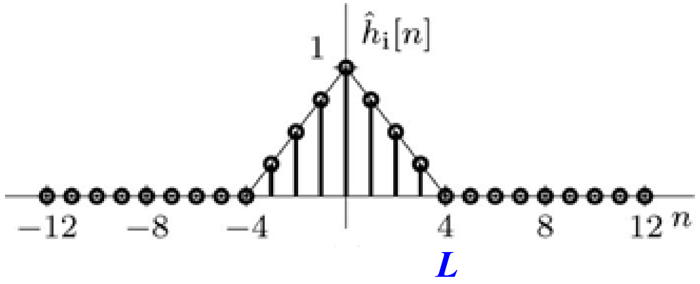
Interpolation



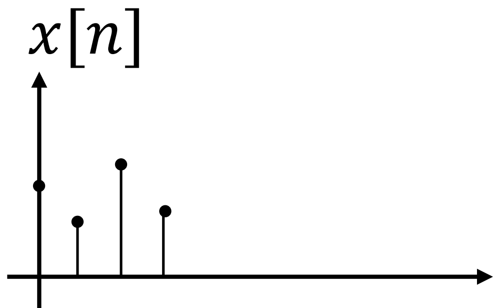
$$h_i[n] = \mathrm{sinc}(n/L)$$



$$\hat{h}_i[n] = \Lambda\left(\frac{n}{2L}\right)$$



$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] h_i[n-kL]$$

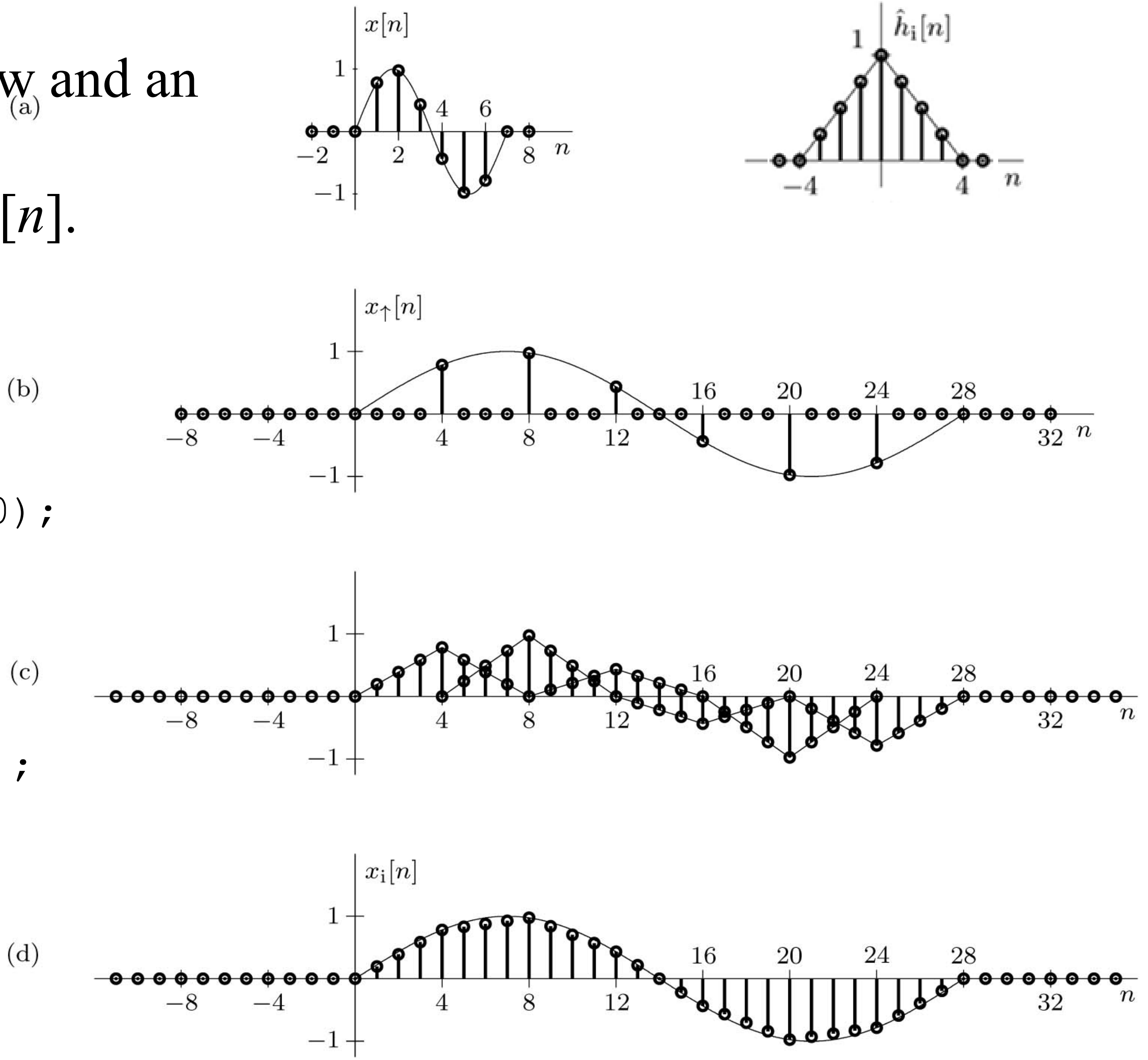


$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\,\mathrm{sinc}\left(\frac{n-kL}{L}\right)$$

Example

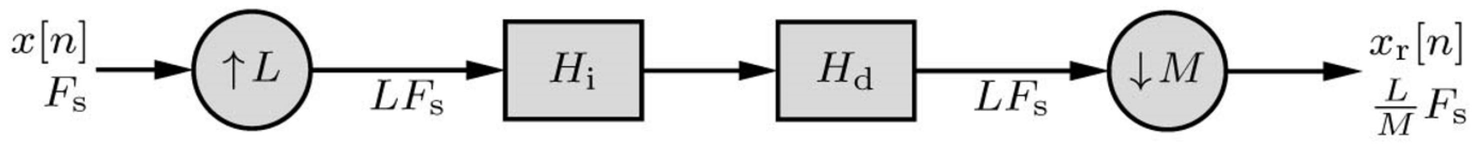
Using the $L = 4$ linear interpolator in Fig. below and an input of $x[n] = \sin(2\pi n/7)(u[n]-u[n-7])$, determine and sketch the interpolator output $x_i[n]$.

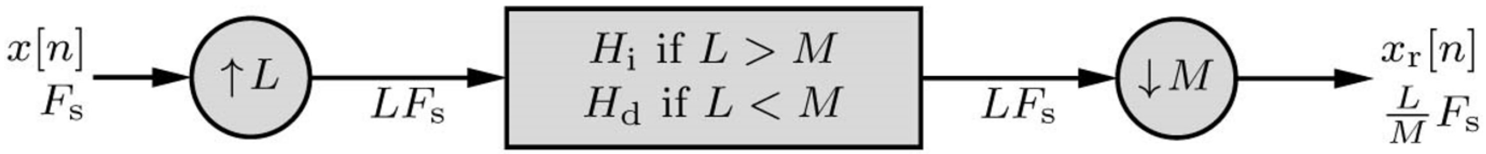
$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] h_i[n - kL] \qquad \hat{h}_i[n] = \Lambda\left(\frac{n}{8}\right)$$




```
x = @(n)
sin(2*pi*n/7).*((n>=0)&(n<7)).*(mod(n,1)==0);
n = -2:8;
subplot(311); stem(n,x(n)); xlabel('n');
ylabel('x[n]');
xup = @(n) x(n/4); n2 = (n(1)*4:n(end)*4);
subplot(312); stem(n2,xup(n2)); xlabel('n');
ylabel('x_{\uparrow}[n]');
hi = [1 2 3 4 3 2 1]/4;
n3 = (-3+n2(1):3+n2(end));
subplot(313); stem(n3,conv(xup(n2),hi));
xlabel('n'); ylabel('x_i[n]');
```

Fractional Sampling Rate Conversion





- An interpolator/decimator cascade used for fractional sampling rate changes.
- Upsampling by L followed by downsampling by M changes the overall sampling rate by a fractional amount L/M.
- The two lowpass filters H_i and H_d, being in cascade, can be replaced by a single lowpass filter of cutoff frequency π/L or π/M, whichever is lower.
- Preferable to do upsampling prior to downsampling to avoid loss of information.

Example: Suppose that the bandwidth of $x[n]$ is $\pi/3$, and we wish to change the sampling rate by factor 3/5.

6.7
Generalization of the DTFT to the z-Transform

Generalization of the DTFT to the z-Transform

1) DTFT synthesize an arbitrary signal $x[n]$ using sinusoids or complex exponentials of the form $e^{j\Omega n}$. DTFT exist only for absolutely summable signals.

2) In System analysis, DTFT is incapable of handling exponentially growing (or decaying) signals.

3) z-transform avoid the limitation of the DTFT by generalizing the complex frequency by replacing $j\Omega$ by $\sigma + j\Omega$, where Ω is the oscillation rate and σ is the decay (or grow) rate.

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

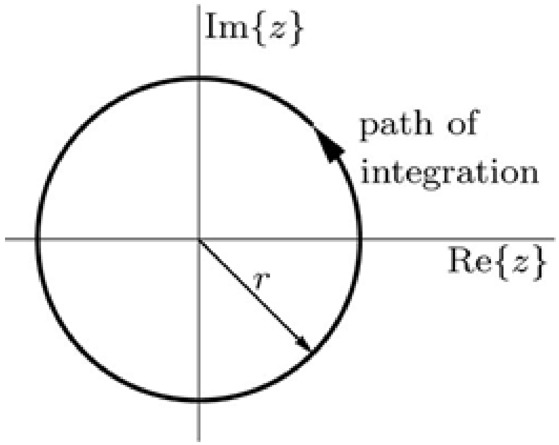
$j\Omega \rightarrow \sigma + j\Omega$

$$\sum_{n=-\infty}^{\infty} x[n]\, e^{-(\sigma+j\Omega)n}$$

$z = e^{\sigma + j\Omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

z-transform



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(j\Omega)\, e^{j\Omega n}\, d\Omega$$

$$x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz$$

Inverse z-transform