ENGR 4333/5333: Digital Signal Processing

Discrete-Time System Analysis Using the z-Transform

Chapter 7

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Outline

- The z-Transform
- The inverse z-Transform
- Properties of the z-Transform
- Z-Transform Solution of Linear Difference Equations
- Block Diagram and System Realization
- Frequency Response of Discrete-Time Systems
- Finite Word-Length Effects
- Connection between the Laplace and z-Transform

The Bilateral z-Transform

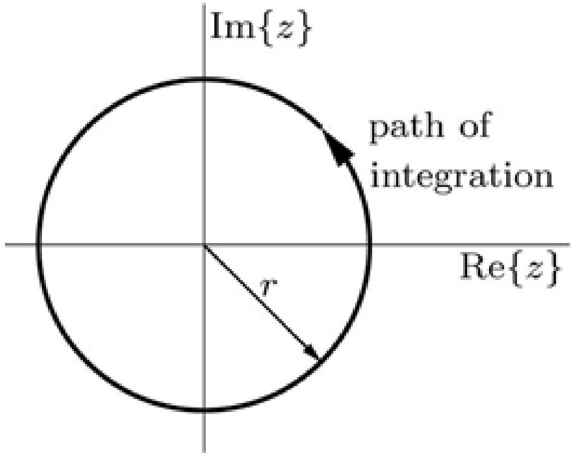
The z-transform is a mathematical tool in system analysis and design. It represent input $x[n]$ as a sum of everlasting exponentials (complex frequency) of the form z^n.

The Laplace Transform $$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

The Bilateral z-transform: $x[n]$ exist for positive and negative n.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad z = e^{\sigma + j\Omega}$$

The inverse z-transform $$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$



$$X(z) = \mathcal{Z}\{x[n]\} \quad \text{and} \quad x[n] = \mathcal{Z}^{-1}\{X(z)\} \qquad x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)$$

Example

Find the z-transform and the corresponding ROC for the causal signal $x[n] = \gamma^n u[n]$.

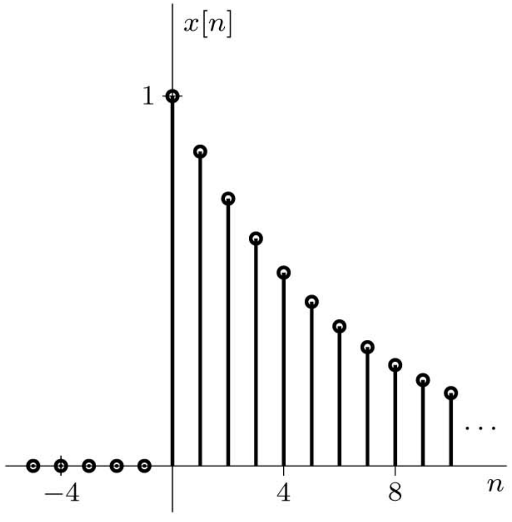
$$\sum_{m=p}^{n} r^m = \frac{r^p - r^{n+1}}{1-r}$$

Solution

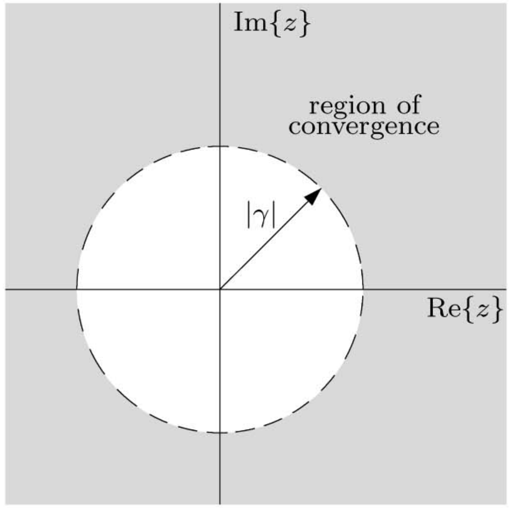
$$X(z) = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n = 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

$$X(z) = \frac{1}{1 - \frac{\gamma}{z}} = \frac{z}{z-\gamma},$$

$$\text{ROC:} \quad \left|\frac{\gamma}{z}\right| < 1 \quad \text{or} \quad |z| > |\gamma|$$



(a)



(b)

Note: for causal signal the ROC extend outward to the centered circle.

Example

Find the z-transform and the corresponding ROC for the causal signal $y[n] = -\gamma^n u[-n-1]$.

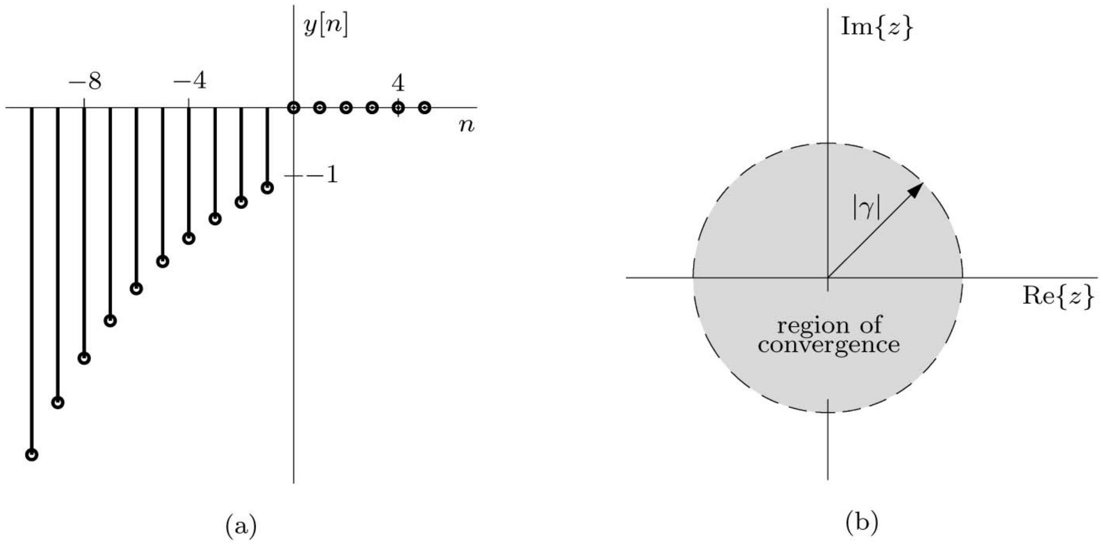
$$\sum_{m=p}^{n} r^m = \frac{r^p - r^{n+1}}{1-r}$$

Solution

$$Y(z) = \sum_{n=-\infty}^{-1} -\left(\frac{\gamma}{z}\right)^n$$

$$Y(z) = \frac{z}{z-\gamma}$$

$$\text{ROC:} \quad \left|\frac{z}{\gamma}\right| < 1 \quad \text{or} \quad |z| < |\gamma|$$



Note: for anti-causal signal the ROC extend inward to the centered circle.

Existence of the Bilateral z-Transform

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

$$|X(z)| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z|^n} < \infty$$

- The z-transform exist for any signal $x[n]$ that grows no faster than an exponential signal r^n, for some real and positive r.
- For a finite length sequence the ROC is the entire z-plane except zero if $x[n]$ is nonzero for some positive n, and ∞ if $x[n]$ is nonzero for some negative n.
- If $z = x + y$ then the region of convergence for z (R_z) is at least $R_x \cap R_y$.

Example

Find the z-transform of $w[n] = x[n] + y[n]$ where $x[n] = (0.9)^n u[n]$ and $y[n] = (1.2)^n u[-n-1]$. What is the z-transform of $w[n]$ if $x[n]$ is changed to $(2)^n u[n]$.

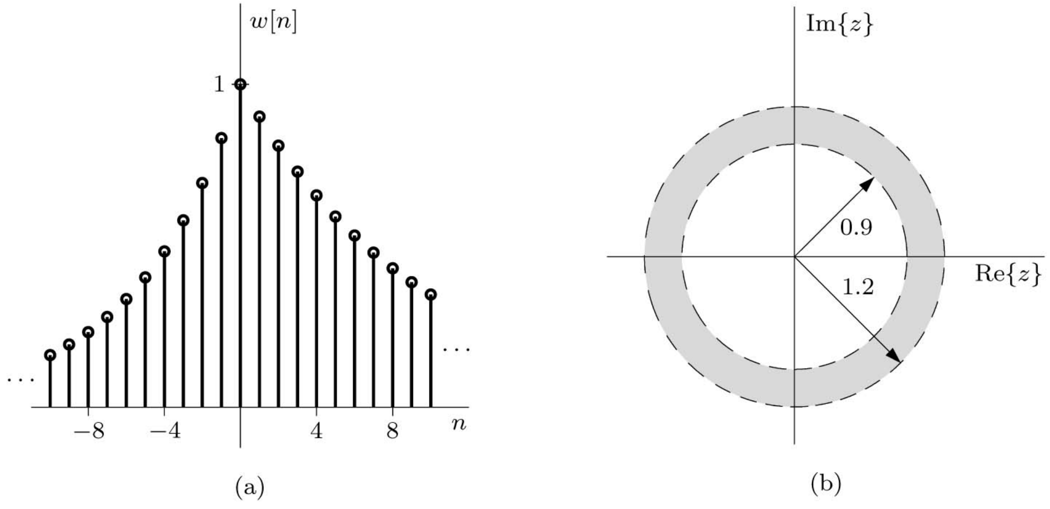
Solution

$$X(z) = \frac{z}{z-0.9} \quad \text{with ROC } |z| > 0.9$$

$$Y(z) = \frac{-z}{z-1.2} \quad \text{with ROC } |z| < 1.2$$

$$W(z) = X(z) + Y(z)$$

$$= \frac{z}{z-0.9} + \frac{-z}{z-1.2} = \frac{-0.3z}{(z-0.9)(z-1.2)}$$



ROC: $0.9 < |z| < 1.2$

If $x[n]$ is changed to $(2)^n u[n]$ then the ROC of $X(z)$ becomes $|z| > 2$ and there is no common ROC for X and Y so the z-transform for w does not exist.

The Unilateral z-Transform

When the signal is causal, $x[n] = 0$ for $n < 0$, then the z-transform is unilateral:

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

Example: Determine the unilateral z-transforms of **(a)** $x_a[n] = \delta[n]$, **(b)** $x_b[n] = u[n]$, **(c)** $x_c[n] = \cos(\beta n)u[n]$, and **(d)** $x_d[n] = u[n] - u[n-5]$.

$$X_{\mathrm{a}}(z) = 1, \qquad \text{ROC: all } z$$

$$X_{\mathrm{b}}(z) = \frac{z}{z-1}, \qquad \text{ROC: } \left|\frac{1}{z}\right| < 1 \text{ or } |z| > 1$$

$$X_{\mathrm{c}}(z) = \frac{z[z - \cos(\beta)]}{z^2 - 2\cos(\beta)z + 1}, \qquad \text{ROC: } |z| > 1$$

$$X_{\mathrm{d}}(z) = \frac{z}{z-1}(1 - z^{-5}), \qquad \text{ROC: all } z \neq 0$$

The transfer function H(z) is the z-transform of the impulse response $h[n]$ of an LTID system;

$$h[n] \overset{\mathcal{Z}}{\Longleftrightarrow} H(z) \text{ with ROC } R_h$$

	$x[n]$	$X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
3.	$\gamma^n u[n]$	$\frac{z}{z-\gamma}$	$\lvert z\rvert > \lvert\gamma\rvert$
4.	$\gamma^{n-1}u[n-1]$	$\frac{1}{z-\gamma}$	$\lvert z\rvert > \lvert\gamma\rvert$
5.	$n\gamma^n u[n]$	$\frac{\gamma z}{(z-\gamma)^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
6.	$n^2\gamma^n u[n]$	$\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$	$\lvert z\rvert > \lvert\gamma\rvert$
7.	$\frac{n!}{(n-m)!m!}\gamma^{n-m}u[n]$	$\frac{z}{(z-\gamma)^{m+1}}$	$\lvert z\rvert > \lvert\gamma\rvert$
8.	$\lvert\gamma\rvert^n \cos(\beta n)u[n]$	$\frac{z[z-\lvert\gamma\rvert\cos(\beta)]}{z^2-2\lvert\gamma\rvert\cos(\beta)z+\lvert\gamma\rvert^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
9.	$\lvert\gamma\rvert^n \sin(\beta n)u[n]$	$\frac{z\lvert\gamma\rvert\sin(\beta)}{z^2-2\lvert\gamma\rvert\cos(\beta)z+\lvert\gamma\rvert^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
10.	$\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\frac{z[z\cos(\theta)-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-2\lvert\gamma\rvert\cos(\beta)z+\lvert\gamma\rvert^2}$ $= \frac{(0.5e^{j\theta})z}{z-\lvert\gamma\rvert e^{j\beta}} + \frac{(0.5e^{-j\theta})z}{z-\lvert\gamma\rvert e^{-j\beta}}$	$\lvert z\rvert > \lvert\gamma\rvert$
11.	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$ $r = \sqrt{\frac{a^2\lvert\gamma\rvert^2+b^2-2abc}{\lvert\gamma\rvert^2-c^2}}$ $\beta = \cos^{-1}\left(\frac{-c}{\lvert\gamma\rvert}\right)$ $\theta = \tan^{-1}\left(\frac{ac-b}{a\sqrt{\lvert\gamma\rvert^2-c^2}}\right)$	$\frac{z(az+b)}{z^2+2cz+\lvert\gamma\rvert^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
12.	$\delta[n-k]$	z^{-k}	$\lvert z\rvert > 0 \quad k>0$ $\lvert z\rvert < \infty \quad k<0$
13.	$-u[-n-1]$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
14.	$-\gamma^n u[-n-1]$	$\frac{z}{z-\gamma}$	$\lvert z\rvert < \lvert\gamma\rvert$
15.	$-n\gamma^n u[-n-1]$	$\frac{z\gamma}{(z-\gamma)^2}$	$\lvert z\rvert < \lvert\gamma\rvert$

7.2
The Inverse z-Transform

The Inverse z-Transform

Using the contour integral to find the inverse z-transform require the knowledge of complex variable theory so we will use the table and partial fraction expansions to find the inverse z-transform.

▷ **Example 7.5 (Inverse Unilateral z-Transform by Partial Fraction Expansion)**

Using partial fraction expansions and Table 7.1, determine the inverse unilateral z-transforms of

(a) $X_a(z) = \frac{8z-19}{(z-2)(z-3)}$ (b) $X_b(z) = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$ (c) $X_c(z) = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

$$x_a[n] = \left[3(2)^{n-1} + 5(3)^{n-1}\right] u[n-1]$$

or

$$x_a[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}(2)^n + \frac{5}{3}(3)^n\right] u[n]$$

$$x_b[n] = -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right] u[n]$$

$$x_c[n] = [2 + 3.2016(5)^n \cos(0.9273n - 2.2455)]\, u[n]$$

Read example in Textbook

Example

Using partial fraction expansions and the Table, determine the inverse bilateral *z*-transform of

$$X(z) = \frac{-z(z+0.4)}{(z-0.8)(z-2)}$$

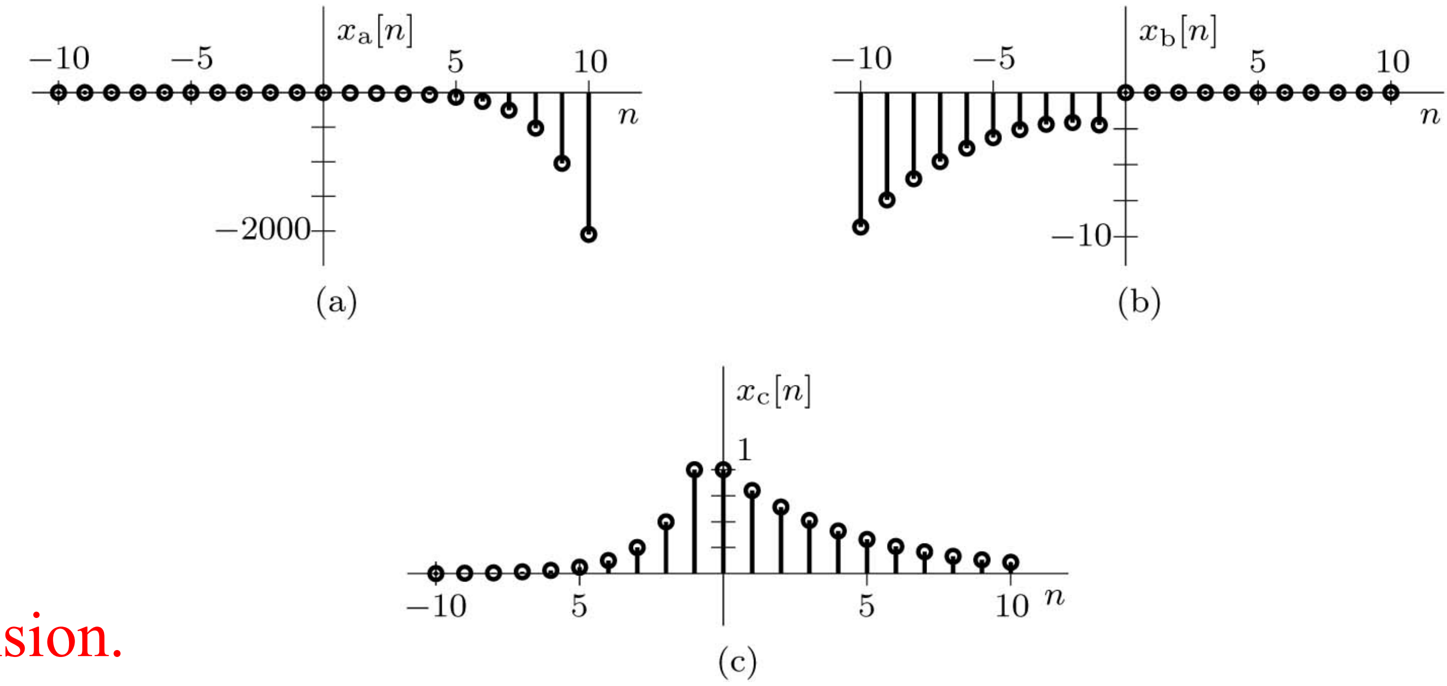
if the ROC is **(a)** $|z| > 2$, **(b)** $|z| < 0.8$, and **(c)** $0.8 < |z| < 2$.

Solution

$$x_a[n] = [(0.8)^n - 2(2)^n]\, u[n]$$

$$x_b[n] = [-(0.8)^n + 2(2)^n]\, u[-n-1]$$

$$x_c[n] = (0.8)^n u[n] + 2(2)^n u[-n-1]$$



Read z-Transform by power series expansion.

7.3
Properties of the z-Transform

Properties of z-Transform

Linear Property

$$ax[n] + by[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} aX(z) + bY(z)$$

Complex-Conjugation Property

$$\text{if } x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z) \text{ with ROC } R_x, \text{then } x^*[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X^*(z^*) \text{ with ROC } R_x$$

Time Scaling Property

$$\text{if } x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z), \text{ then } x_{\uparrow}[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z^L)$$

$$\gamma^n u[n] \longleftrightarrow \frac{z}{z-\gamma}$$

Time-Reversal Property

$$\text{if } x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z) \text{ with ROC } R_x, \text{ then } x[-n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(1/z) \text{ with ROC } 1/R_x$$

Properties of z-Transform

Bilateral *z*-Transform Time-Shifting Property

$$\text{if } x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z) \text{ with ROC } R_x, \text{ then } x[n-m] \overset{\mathcal{Z}}{\Longleftrightarrow} z^{-m}X(z) \text{ with ROC almost } R_x$$

Unilateral *z*-Transform Time-Shifting Property

Right Shift (Delay)

$$x[n-m]u[n-m] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z), \ (m > 0)$$

$$x[n-m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z) + z^{-m}\sum_{n=1}^{m} x[-n]z^{n}, \ (m > 0)$$

$$x[n-1]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-1}X(z) + x[-1].$$

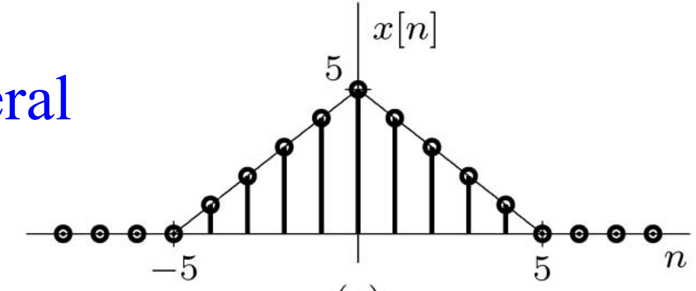
$$x[n-2]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-2}X(z) + z^{-1}x[-1] + x[-2]$$

Left Shift (Advance)

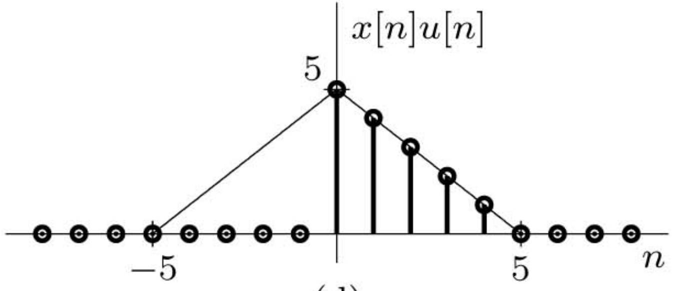
$$x[n+m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{m}X(z) - z^{m}\sum_{n=0}^{m-1} x[n]z^{-n} \qquad (m > 0)$$

Bilateral

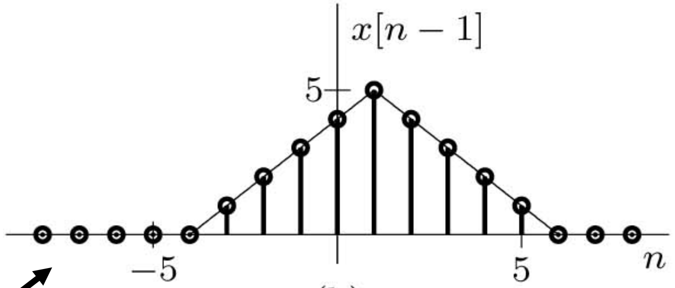
Unilateral



(a)



(d)



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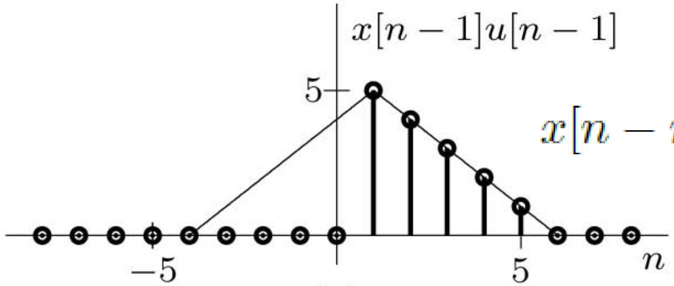


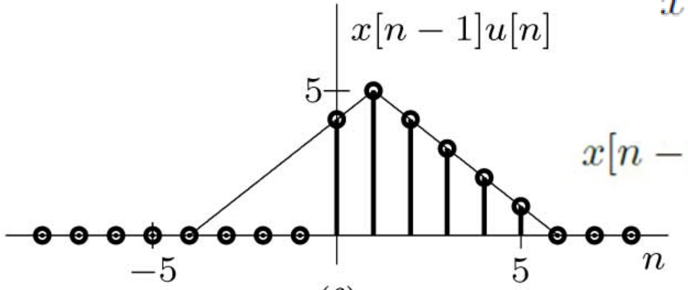
(e)

$$x[n-m]u[n-m] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z), \;\; (m > 0)$$

$$x[n-1]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-1}X(z) + x[-1]$$

if $x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)$ with ROC R_x,

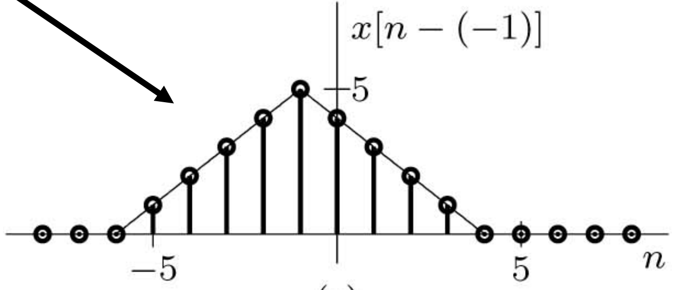
then $x[n-m] \overset{\mathcal{Z}}{\Longleftrightarrow} z^{-m}X(z)$ with ROC almost R_x



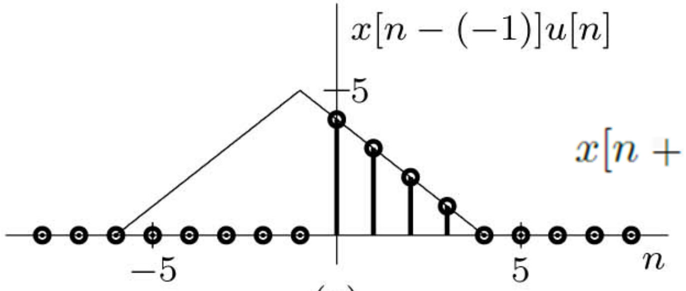
(f)

$(m > 0)$

$$x[n-m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z) + z^{-m}\sum_{n=1}^{m} x[-n]z^{n}$$



(c)



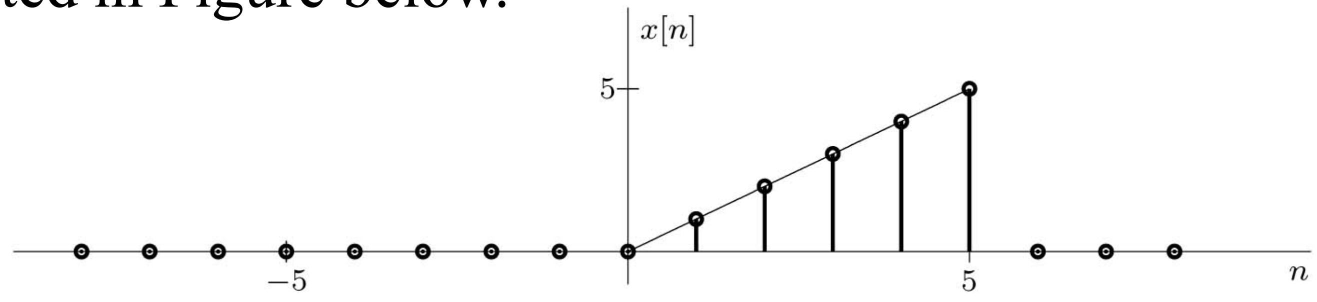
(g)

$(m > 0)$

$$x[n+m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{m}X(z) - z^{m}\sum_{n=0}^{m-1} x[n]z^{-n}$$

Example

Find the z-transform of the signal $x[n]$ depicted in Figure below.



Solution

$$x[n] = nu[n] - \{(n-6)u[n-6] + 6u[n-6]\}$$

$$n\gamma^n u[n] \overset{Z}{\Longleftrightarrow} \frac{\gamma z}{(z-\gamma)^2}$$

$$X(z) = \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)}$$

$$X(z) = \frac{z^6 - 6z + 5}{z^5(z-1)^2}$$

Properties of z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$u[n]$	$\longleftrightarrow$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\gamma^n u[n]$	$\longleftrightarrow$	$\frac{z}{z-\gamma}$	$\lvert z\rvert > \lvert\gamma\rvert$

Z-Domain Scaling Property

If $x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)$ with ROC R_x,

then $\gamma^n x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X\left(\frac{z}{\gamma}\right)$ with ROC $|\gamma| R_x$

Z-Domain Differentiation Property

If $x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)$ with ROC R_x,

then $nx[n] \overset{\mathcal{Z}}{\Longleftrightarrow} -z\frac{d}{dz}X(z)$ with ROC R_x

Z-Domain Convolution Property

If $x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)$ with ROC R_x and $y[n] \overset{\mathcal{Z}}{\Longleftrightarrow} Y(z)$ with ROC R_y

then $x[n] * y[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)Y(z)$ with ROC at least $R_x \cap R_y$

Properties of z-Transform

Using Convolution to find the system zero-state response

$$y[n] = x[n] * h[n]$$

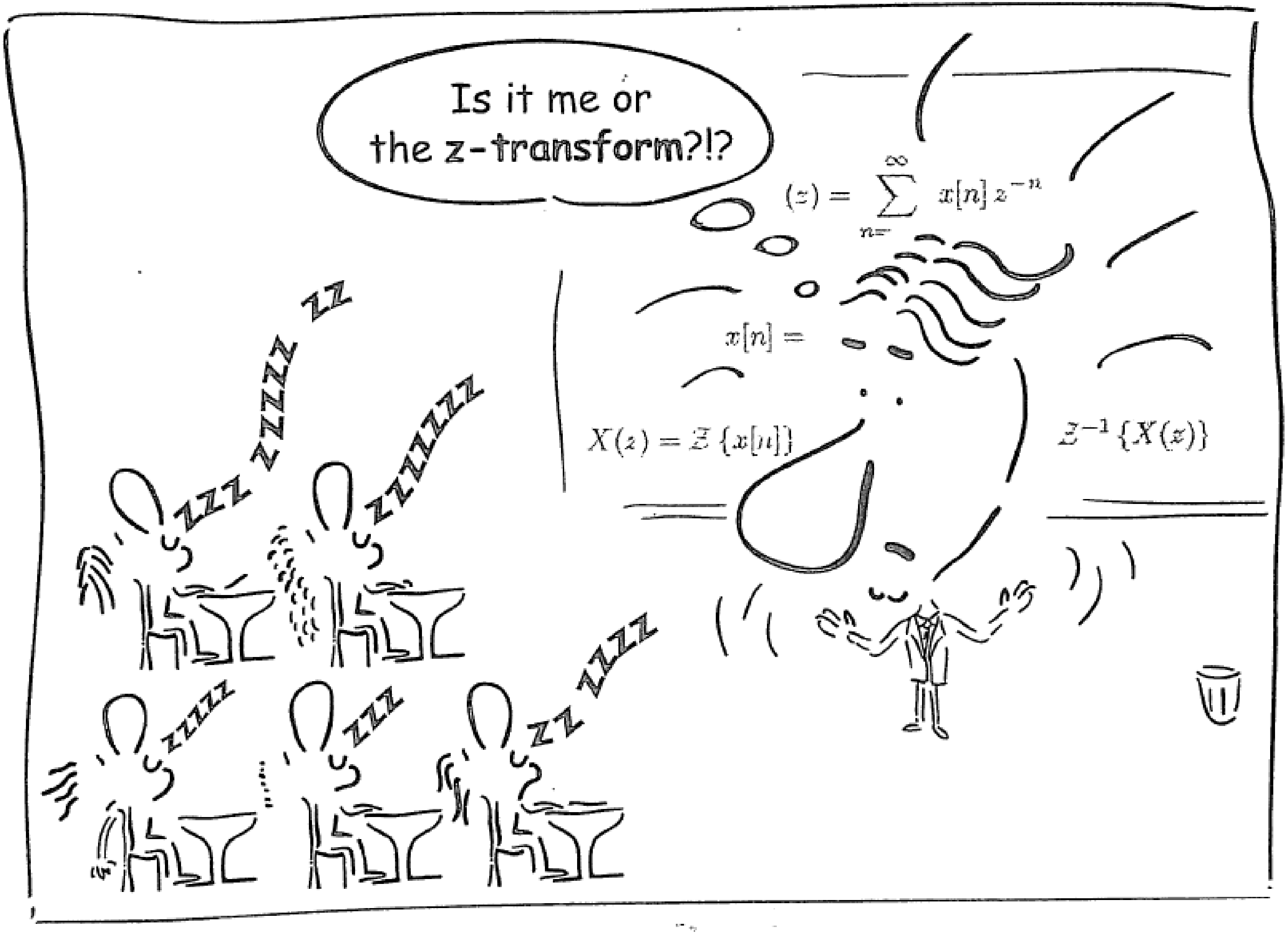
$$Y(z) = X(z)H(z) \text{ with ROC at least } R_x \cap R_h$$

Bilateral z-Transform	Unilateral z-Transform
Synthesis: $x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$	**Synthesis:** $x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$
Analysis: $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$, ROC: R_x	**Analysis:** $X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$
Linearity: $ax[n]+by[n] \overset{\mathcal{Z}}{\Longleftrightarrow} aX(z)+bY(z)$, ROC: At least $R_x \cap R_y$	**Linearity:** $ax[n]+by[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} aX(z)+bY(z)$
Complex Conjugation: $x^*[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X^*(z^*)$, ROC: R_x	**Complex Conjugation:** $x^*[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} X^*(z^*)$
Time Reversal: $x[-n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(1/z)$, ROC: $1/R_x$	**Time Reversal:**
Time Shifting: $x[n-m] \overset{\mathcal{Z}}{\Longleftrightarrow} z^{-m}X(z)$, ROC: Almost R_x	**Time Shifting:** If $m>0$: $x[n-m]u[n-m] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z)$ (general case given below)
z-Domain Scaling: $\gamma^n x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z/\gamma)$, ROC: $\|\gamma\| R_x$	**z-Domain Scaling:** $\gamma^n x[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} X(z/\gamma)$
z-Domain Differentiation: $nx[n] \overset{\mathcal{Z}}{\Longleftrightarrow} -z\frac{d}{dz}X(z)$, ROC: R_x	**z-Domain Differentiation:** $nx[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} -z\frac{d}{dz}X(z)$
Time Convolution: $x[n] * y[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)Y(z)$, ROC: At least $R_x \cap R_y$	**Time Convolution:** $x[n] * y[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} X(z)Y(z)$

Unilateral z-Transform Time Shifting, General Case

If $m > 0$: $x[n-m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z) + z^{-m}\sum_{n=1}^{m} x[-n]z^{n}$

If $m < 0$: $x[n-m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z) - z^{-m}\sum_{n=0}^{-m-1} x[n]z^{-n}$


Is it me or the z-transform?!?
$(z) = \sum_{n=}^{\infty} x[n]\, z^{-n}$
$x[n] =$
$X(z) = \mathcal{Z}\{x[n]\}$
$\mathcal{Z}^{-1}\{X(z)\}$
ZZZ ZZZZZ ZZ
ZZZZZZZ
ZZZZZ
ZZZ
ZZ ZZZZ

7.4
z-Transform Solution of Linear Difference Equations

Example: Total Response

When initial conditions are present, the unilateral z-transform is generally the appropriate analysis tool. When only the zero-state response is required, either the bilateral or unilateral z-transform may be appropriate.

Example: Given input $x[n] = (0.5)^n u[n]$ and initial conditions $y[-1] = 11/6$ and $y[-2] = 37/36$, use the unilateral z-transform to solve the second-order ($K = 2$) constant-coefficient linear difference equation

$$y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n].$$

Solution

$$\frac{Y(z)}{z} = \frac{(3z^2 - 9.5z + 10.5)}{(z-0.5)(z^2-5z+6)}$$

$$Y(z) = \frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{7}{3}\left(\frac{z}{z-2}\right) + \frac{18}{5}\left(\frac{z}{z-3}\right)$$

$$y[n] = \left[\frac{26}{15}(0.5)^n - \frac{7}{3}(2)^n + \frac{18}{5}(3)^n\right] u[n]$$

$$x[n+m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^m X(z) - z^m \sum_{n=0}^{m-1} x[n] z^{-n} \quad (m > 0)$$

Example: Zero-Input and Zero-State Components

Example: Given input $x[n] = (0.5)^n u[n]$ and initial conditions $y[-1] = 11/6$ and $y[-2] = 37/36$, use the unilateral z-transform to solve the second-order ($K = 2$) constant-coefficient linear difference equation

$$y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n].$$

Solution

$$\left(1 - 5z^{-1} + 6z^{-2}\right) Y(z) \underbrace{- \left(3 - 11z^{-1}\right)}_{\text{IC terms}} = \underbrace{\frac{3}{z-0.5} + \frac{5}{z(z-0.5)}}_{\text{input terms}}$$

$$\left(z^2 - 5z + 6\right) Y(z) = \underbrace{z\,(3z - 11)}_{\text{IC terms}} + \underbrace{\frac{z(3z+5)}{(z-0.5)}}_{\text{input terms}} \qquad Y(z) = \underbrace{\frac{z(3z-11)}{z^2 - 5z + 6}}_{\text{ZIR}} + \underbrace{\frac{z(3z+5)}{(z-0.5)(z^2-5z+6)}}_{\text{ZSR}}$$

$$y[n] = \underbrace{\left[5(2)^n - 2(3)^n\right] u[n]}_{\text{ZIR}} + \underbrace{\left[\frac{26}{15}(0.5)^n - \frac{22}{3}(2)^n + \frac{28}{5}(3)^n\right] u[n]}_{\text{ZSR}}$$

The Transfer Function and ZSR of LTID

The Difference Equation of a System

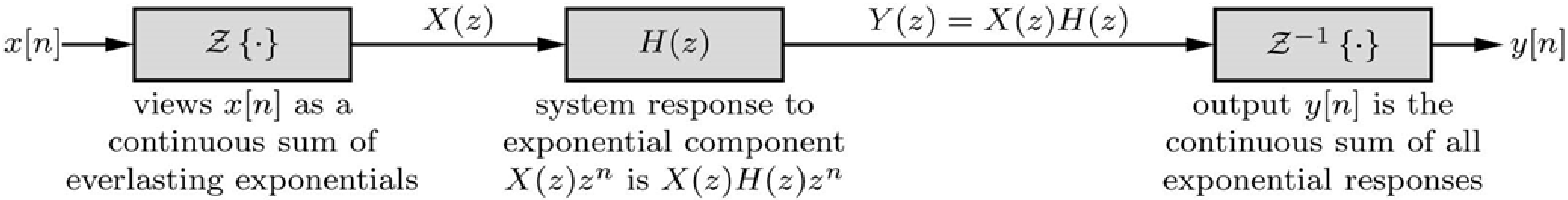
$$y[n+K] + a_1 y[n+(K-1)] + \cdots + a_{K-1} y[n+1] + a_K y[n] = b_0 x[n+K] + b_1 x[n+(K-1)] + \cdots + b_{K-1} x[n+1] + b_K x[n]$$

Take the z-transform, shifting property, and setting all IC to zero the above equation will be

$$\left(z^K + a_1 z^{K-1} + \cdots + a_{K-1} z + a_K\right) Y(z) = \left(b_0 z^K + b_1 z^{K-1} + \cdots + b_{K-1} z + b_K\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \frac{b_0 z^K + b_1 z^{K-1} + \cdots + b_{K-1} z + b_K}{z^K + a_1 z^{K-1} + \cdots + a_{K-1} z + a_K}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}\{\text{zero-state response}\}}{\mathcal{Z}\{\text{input}\}}$$



Example for Zero-State Response

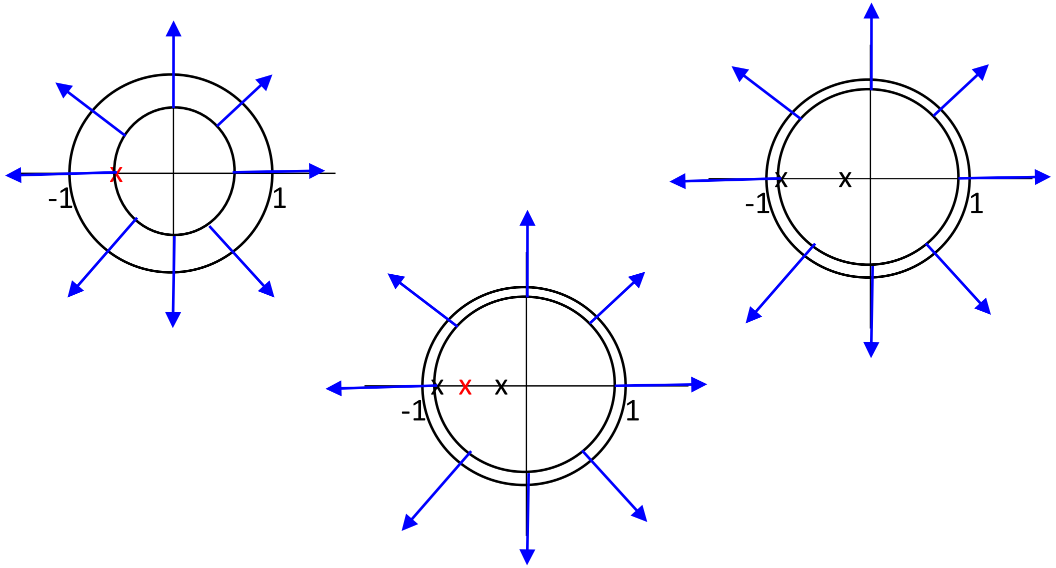
Example: Given input $x[n] = (-2)^{-n}u[n]$, use the z-transform to determine the zero-state response $y[n]$ of a causal LTID system described by the difference equation

$$y[n+2] + y[n+1] + 0.16y[n] = x[n+1] + 0.32x[n].$$

Solution ROC ?

$$H(z) = \frac{B(z)}{A(z)} = \frac{z+0.32}{z^2+z+0.16} \qquad X(z) = \frac{z}{z+0.5}$$

$$Y(z) = X(z)H(z) = \frac{z(z+0.32)}{(z^2+z+0.16)(z+0.5)}$$



$$Y(z) = \frac{2}{3}\left(\frac{z}{z+0.2}\right) - \frac{8}{3}\left(\frac{z}{z+0.8}\right) + 2\left(\frac{z}{z+0.5}\right)$$

$$y[n] = \left[\frac{2}{3}(-0.2)^n - \frac{8}{3}(-0.8)^n + 2(-0.5)^n\right]u[n]$$

Example with Causal and Non-causal Inputs

Example 3: Given two-sided input $x[n] = (0.8)^n u[n] + 2(2)^n u[-n - 1]$, use the z-transform to determine the zero-state response $y[n]$ of a causal LTID system described by the transfer function $H(z) = z / (z - 0.5)$.

Solution

$$\mathcal{Z}\{(0.8)^n u[n]\} = \frac{z}{z-0.8} \quad \text{with ROC } |z| > 0.8$$

$$\mathcal{Z}\{2(2)^n u[-n-1]\} = -\frac{2z}{z-2} \quad \text{with ROC } |z| < 2$$

$$X(z) = \frac{-z(z+0.4)}{(z-0.8)(z-2)}, \qquad \text{ROC:} \quad 0.8 < |z| < 2$$

$$Y(z) = X(z)H(z) = \frac{-z^2(z+0.4)}{(z-0.5)(z-0.8)(z-2)}, \qquad \text{ROC:} \quad 0.8 < |z| < 2$$

$$Y(z) = -\left(\frac{z}{z-0.5}\right) + \frac{8}{3}\left(\frac{z}{z-0.8}\right) - \frac{8}{3}\left(\frac{z}{z-2}\right) \quad \text{with ROC } 0.8 < |z| < 2$$

$$y[n] = \left[-(0.5)^n + \frac{8}{3}(0.8)^n\right] u[n] + \frac{8}{3}(2)^n u[-n-1]$$

Read Example

Example with Inputs with Disjoint ROC

Example 4: For the system $H(z) = z / (z - 0.5)$, find the zero-state response to the input

$$x[n] = \underbrace{(0.8)^n u[n]}_{x_1[n]} + \underbrace{(0.6)^n u[-n-1]}_{x_2[n]}.$$

Solution

$$X_1(z) = \frac{z}{z-0.8} \quad \text{with ROC } |z| > 0.8 \qquad X_2(z) = -\frac{z}{z-0.6} \quad \text{with ROC } |z| < 0.6$$

There is no common ROC between $x_1[n]$ and $x_2[n]$ so use superposition to find the system response to both inputs separately.

$$Y_1(z) = \frac{z^2}{(z-0.5)(z-0.8)} = -\frac{5}{3}\left(\frac{z}{z-0.5}\right) + \frac{8}{3}\left(\frac{z}{z-0.8}\right) \quad \text{with ROC } |z| > 0.8$$

$$Y_2(z) = \frac{-z^2}{(z-0.5)(z-0.6)} = 5\left(\frac{z}{z-0.5}\right) - 6\left(\frac{z}{z-0.6}\right) \quad \text{with ROC } 0.5 < |z| < 0.6$$

$$y_1[n] = \left[-\frac{5}{3}(0.5)^n + \frac{8}{3}(0.8)^n\right] u[n] \qquad y_2[n] = 5(0.5)^n u[n] + 6(0.6)^n u[-n-1]$$

Read Example

System Stability and the Transfer Function $H(z)$

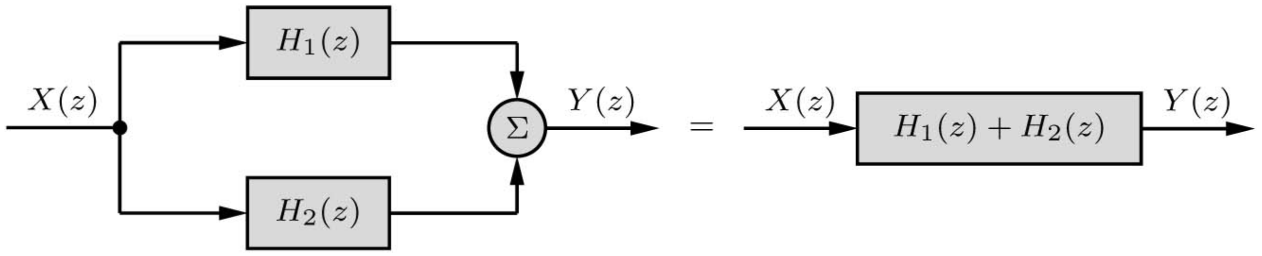
- A causal LTID system is asymptotically stable if and only if all the characteristic roots are inside the unit circle. The roots may be simple or repeated.
- A causal LTID system is marginally stable if and only if there are no roots outside the unit circle and there are non-repeated roots on the unit circle.
- A causal LTID system is unstable if and only if at least one root is outside the unit circle, there are repeated roots on the unit circle, or both.

7.5
Block Diagrams and System Realization

Basic Connections

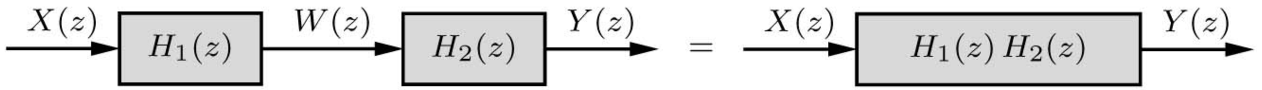
The following is true if there is no loading effect between connected subsystems.

Parallel Connection



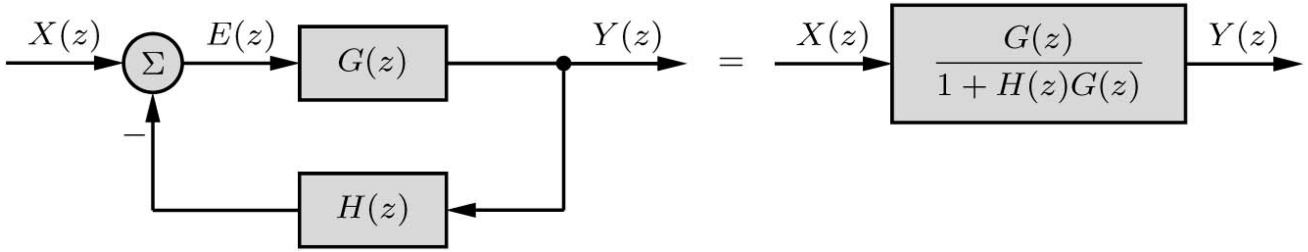
(a)

Cascade Connection



(b)

Feedback System



Direct Form Realization

$$H(z) = \frac{b_0 z^L + b_1 z^{L-1} + \cdots + b_{L-1} z + b_L}{z^K + a_1 z^{K-1} + \cdots + a_{K-1} z + a_K}$$

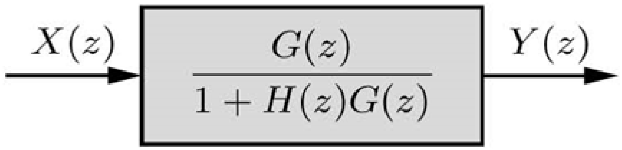
For causal systems, $L \leq K$. Multiply numerator and denominator by z^{-K}. Next express $H(z)$ as a cascade of two systems.

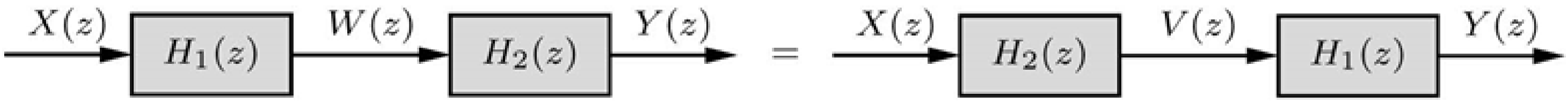
Example:

$$H(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \qquad H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$H(z) = \underbrace{\left(b_0 + b_1 z^{-1} + b_2 z^{-2}\right)}_{H_1(z)} \underbrace{\left(\frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}\right)}_{H_2(z)}$$

Feedback $H_2(z)$

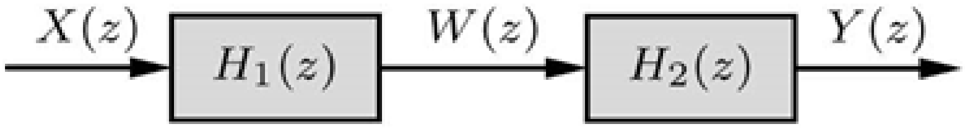




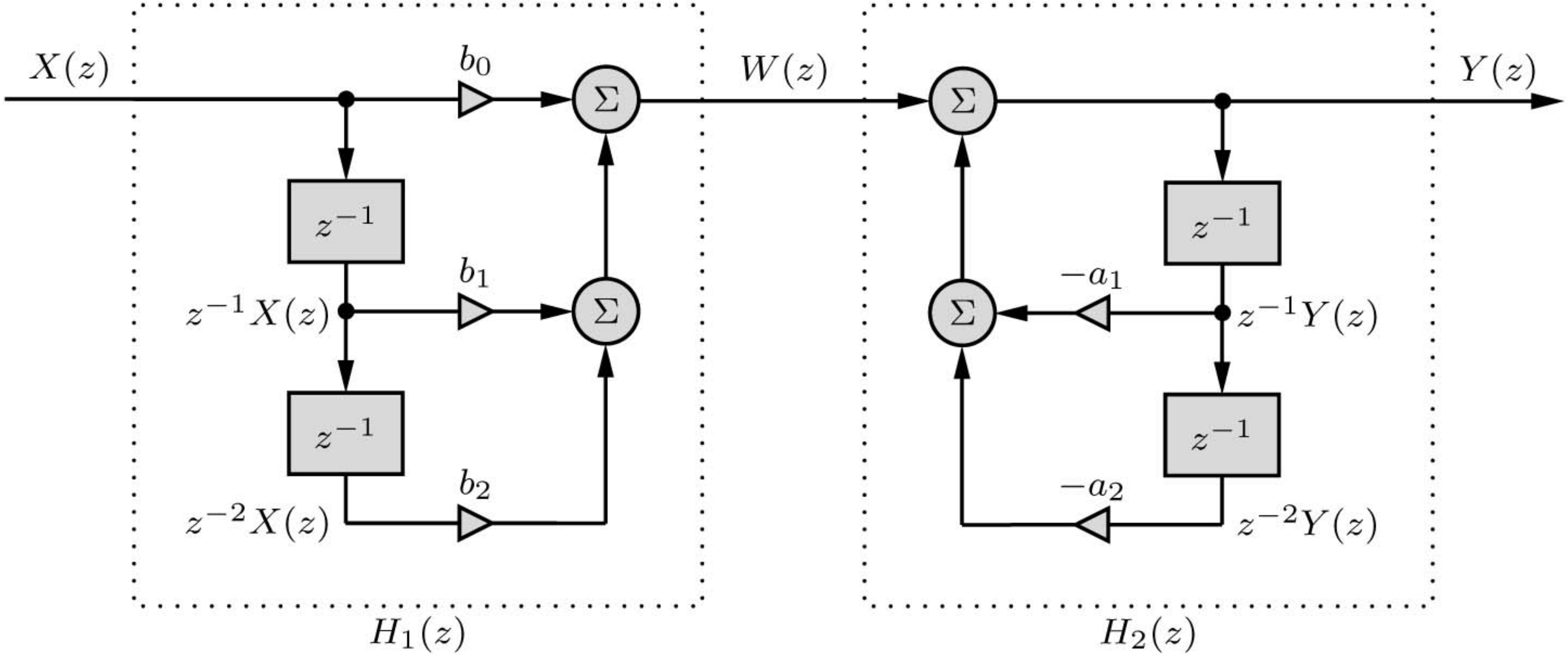
Direct Form I | Direct Form II

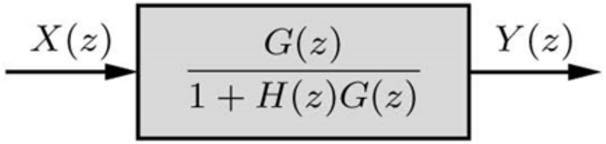
Direct Form I Realization

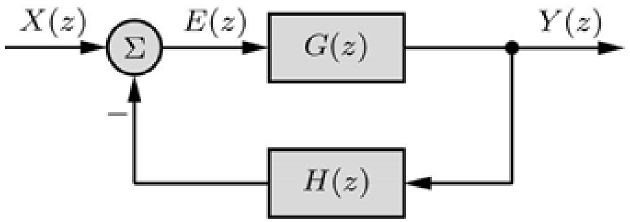
$$H(z) = \underbrace{\left(b_0 + b_1 z^{-1} + b_2 z^{-2}\right)}_{H_1(z)} \underbrace{\left(\frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}\right)}_{H_2(z)}$$



$$Y(z) = W(z) - \left(a_1 z^{-1} + a_2 z^{-2}\right) Y(z)$$

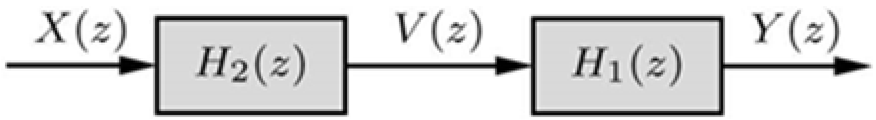




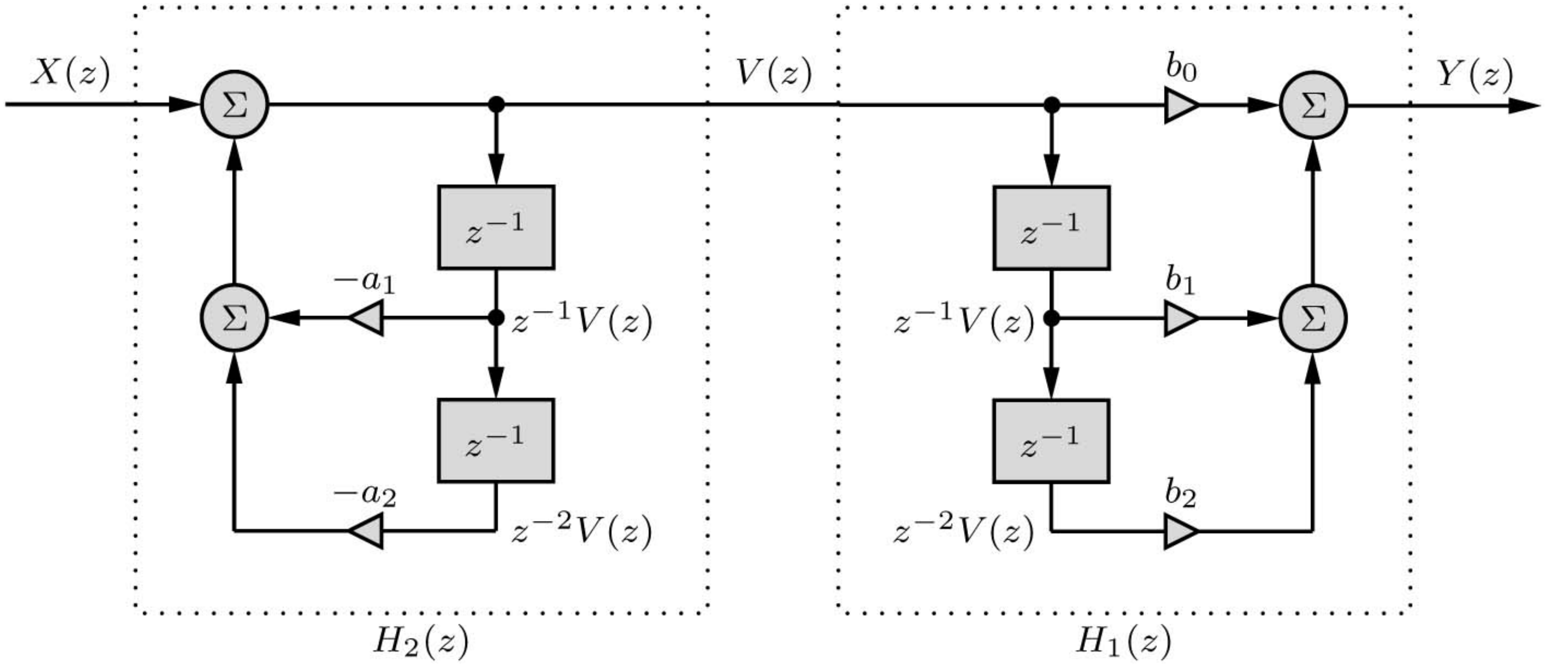


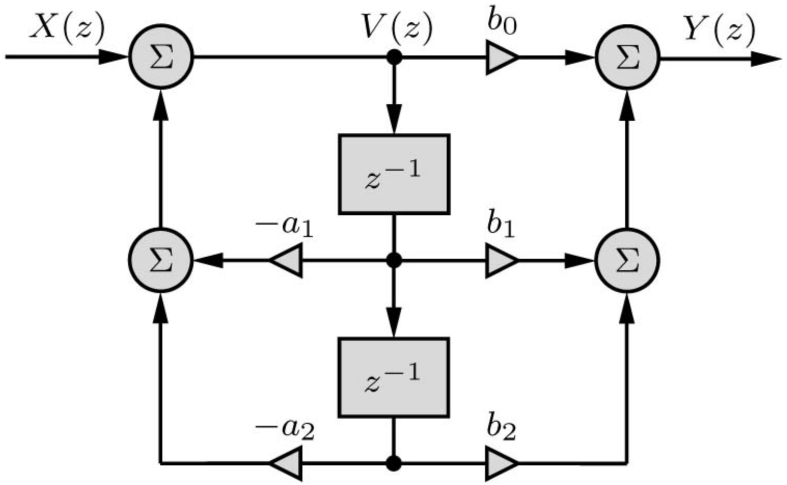
Direct Form II Realization

$$H(z) = \underbrace{\left(\frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}\right)}_{H_2(z)} \underbrace{\left(b_0 + b_1 z^{-1} + b_2 z^{-2}\right)}_{H_1(z)}$$



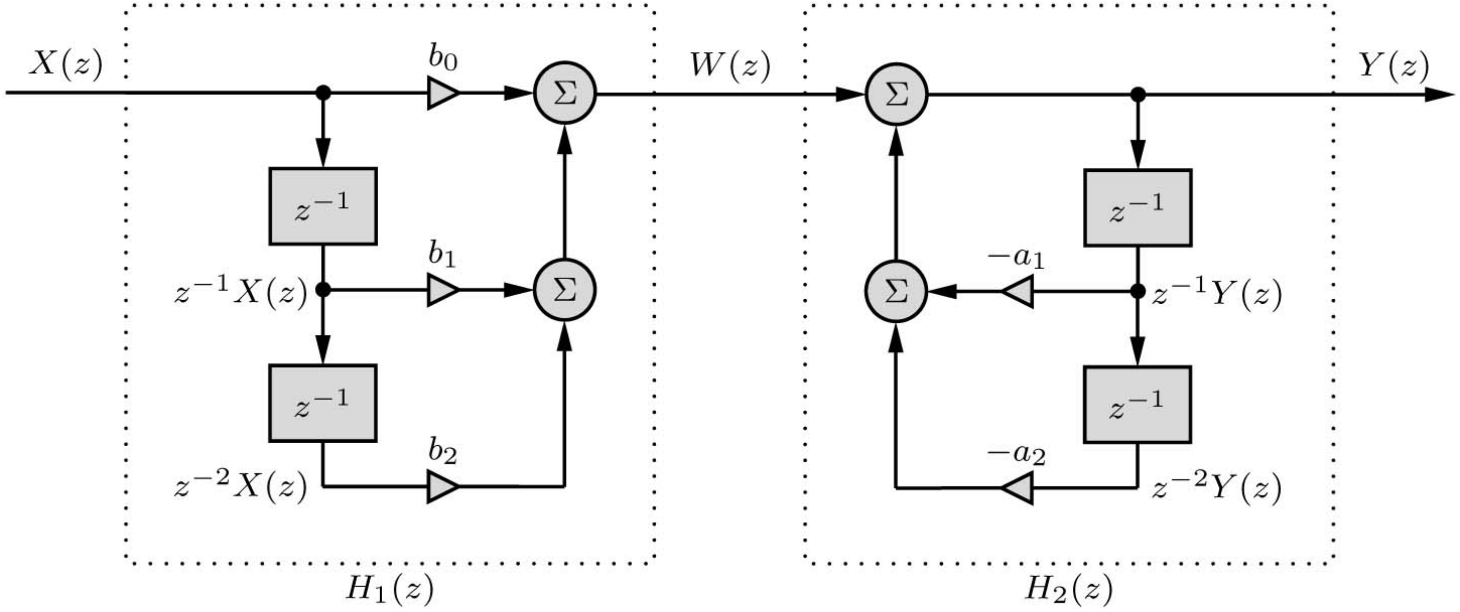
$$V(z) = X(z) - \left(a_1 z^{-1} + a_2 z^{-2}\right) V(z)$$



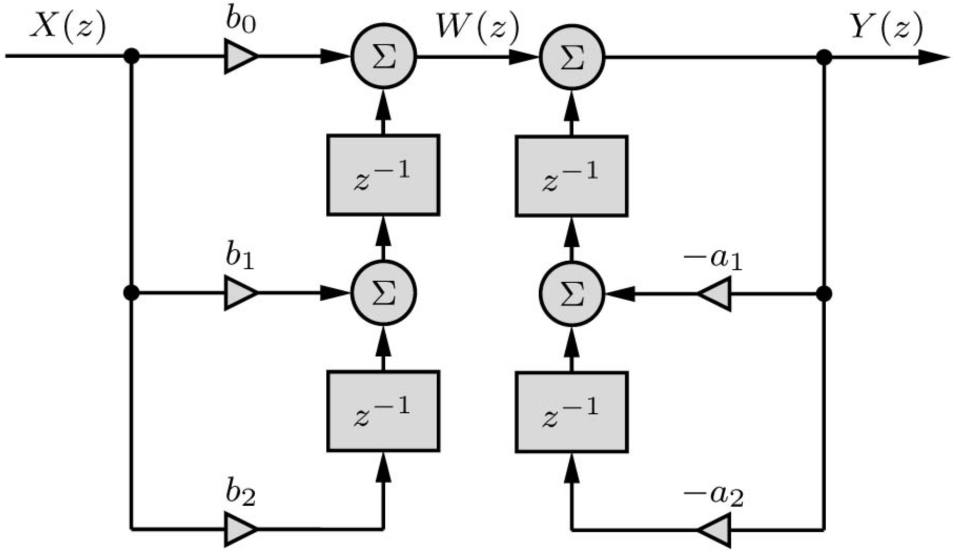


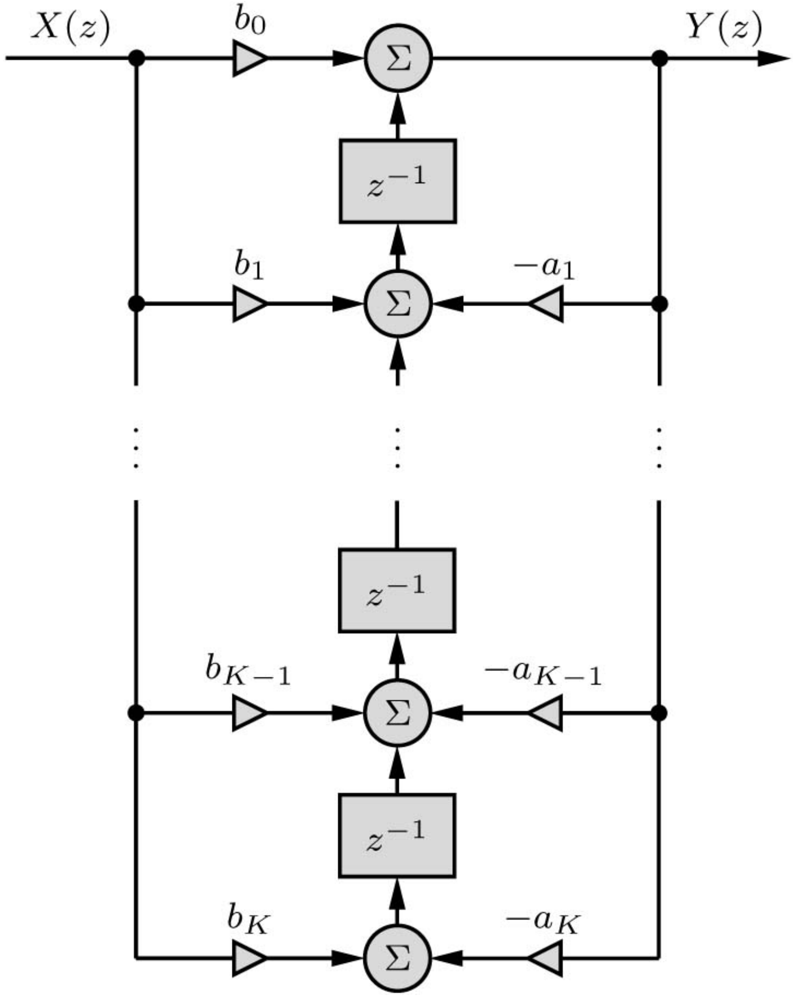
DFII is canonic since the number of delays equal to the order of the transfer function.

Transpose Realization (TDFII)



$$H(z) = \frac{b_0 z^L + b_1 z^{L-1} + \cdots + b_{L-1} z + b_L}{z^K + a_1 z^{K-1} + \cdots + a_{K-1} z + a_K}$$

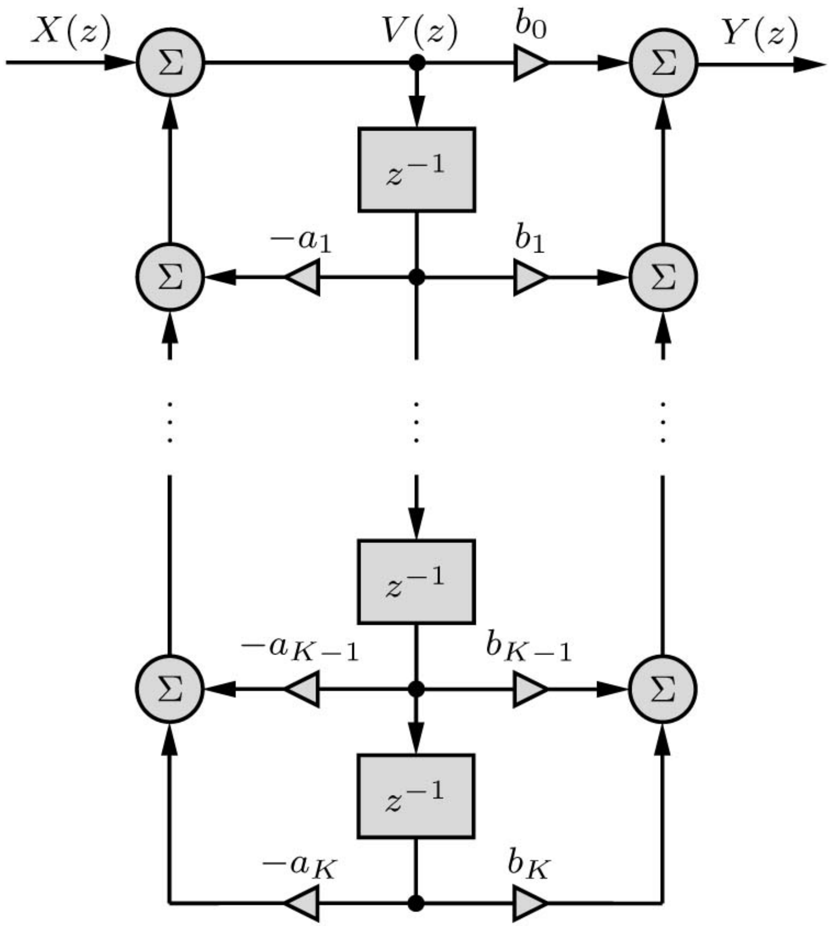




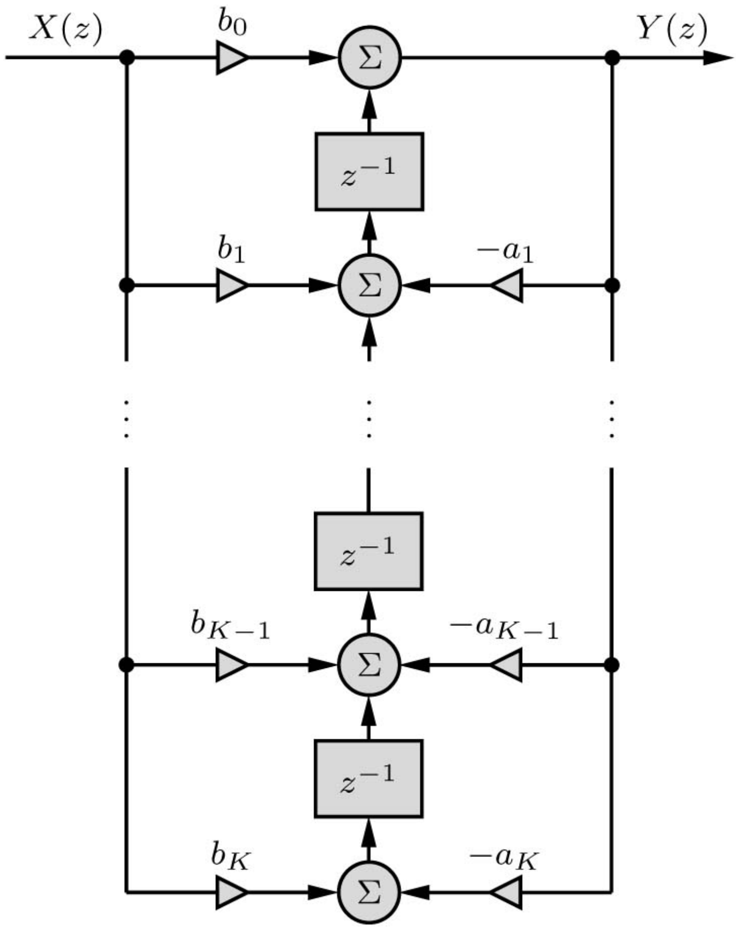
DFII and TDFII

$$H(z) = \frac{b_0 z^L + b_1 z^{L-1} + \cdots + b_{L-1} z + b_L}{z^K + a_1 z^{K-1} + \cdots + a_{K-1} z + a_K}$$

Direct Form II (DFII)



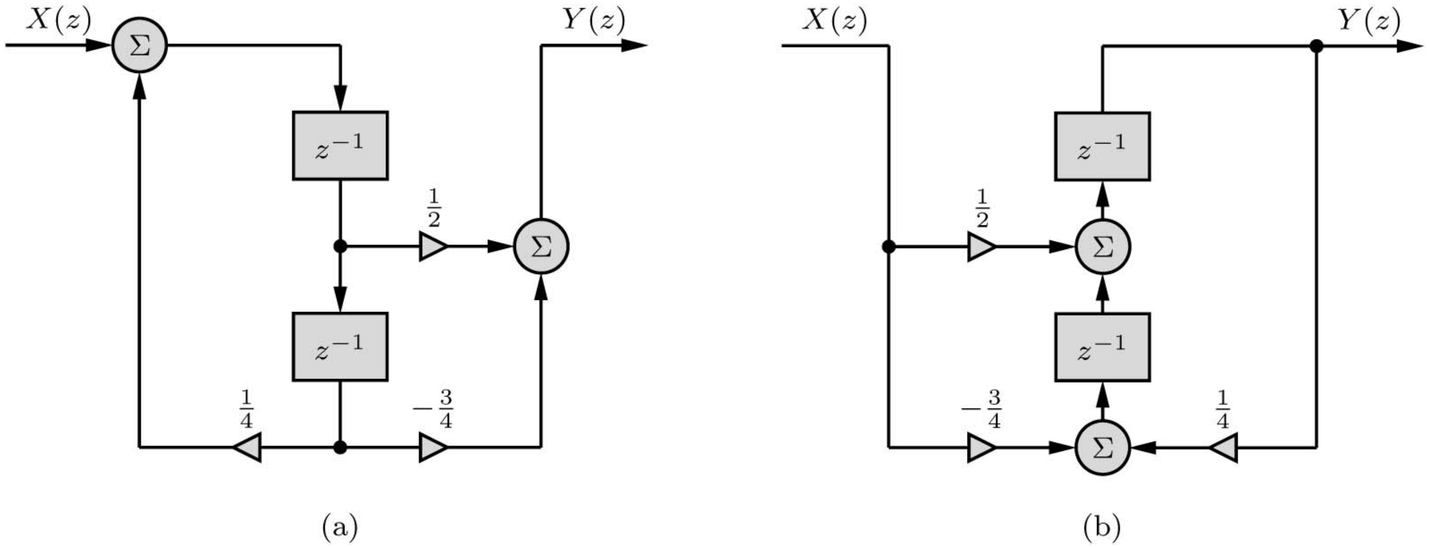
Transpose Direct Form II (TDFII)



Example

Find the DFII and TDFII realizations of an LTID system with transfer function

$$H(z) = \frac{2z - 3}{4z^2 - 1}$$



Direct Form II

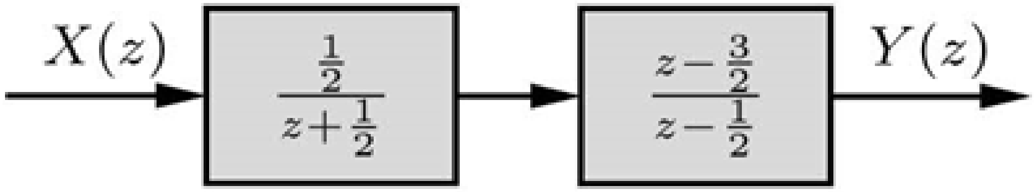
Transpose Direct Form II

Cascade and Parallel Realization

$$H(z) = \frac{2z-3}{4z^2-1} = \frac{1/2(z-3/2)}{(z+1/2)(z-1/2)}$$

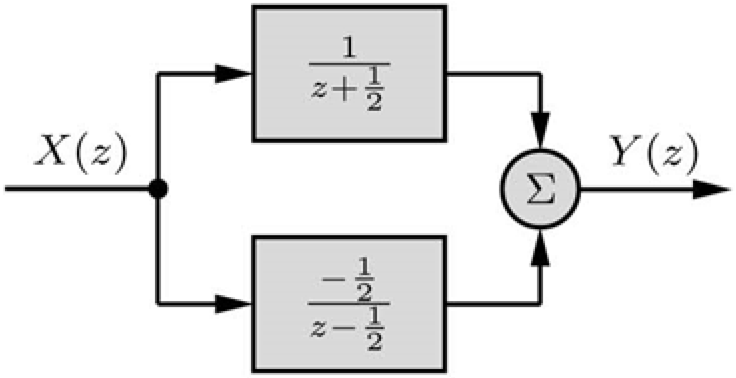
Cascade

$$H(z) = \underbrace{\left(\frac{1/2}{z+1/2}\right)}_{H_1(z)}\underbrace{\left(\frac{z-3/2}{z-1/2}\right)}_{H_2(z)}$$



Parallel

$$H(z) = \underbrace{\frac{1}{z+1/2}}_{H_3(z)} + \underbrace{\frac{-1/2}{z-1/2}}_{H_4(z)}$$



Realization of Complex-Conjugate Roots

$$H(z) = \frac{z^3 + z}{16z^3 - 28z^2 + 20z - 6}$$

$$H(z) = \left(\frac{\frac{1}{16}z}{z - \frac{3}{4}}\right)\left(\frac{z + j}{z - (\frac{1}{2} + j\frac{1}{2})}\right)\left(\frac{z - j}{z - (\frac{1}{2} - j\frac{1}{2})}\right)$$

Cascade

$$H(z) = \left(\frac{\frac{1}{16}z}{z - \frac{3}{4}}\right)\left(\frac{z^2 + 1}{z^2 - z + \frac{1}{2}}\right)$$

Parallel

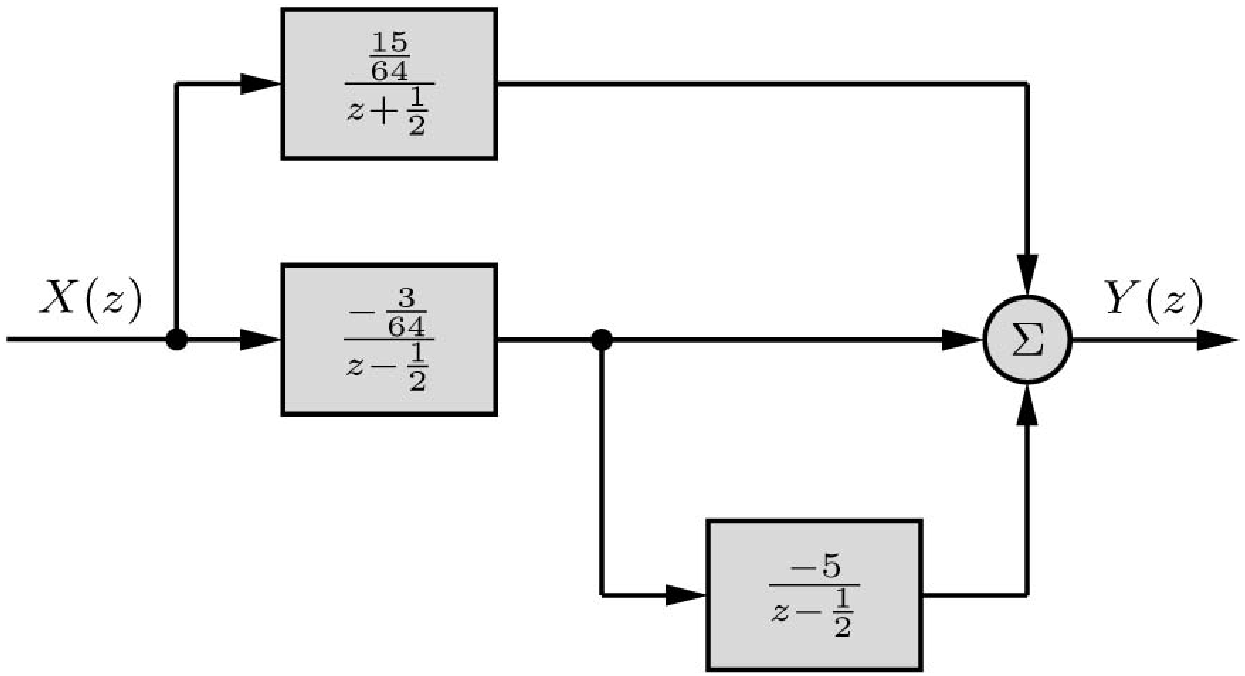
$$H(z) = \frac{\frac{5}{16}z}{z - \frac{3}{4}} + \frac{-\frac{1}{4}z^2 + \frac{1}{8}z}{z^2 - z + \frac{1}{2}}$$

Realization of Repeated Roots

For parallel connection you need to reuse some of the blocks so the number of delay equal the order of the transfer function.

Example: Determine a parallel realization of $H(z) = \dfrac{\frac{3}{16}(z^2+1)}{(z+1/2)(z-1/2)^2}$

$$H(z) = \frac{-\frac{3}{64}}{z-\frac{1}{2}} + \frac{\frac{15}{64}}{(z-\frac{1}{2})^2} + \frac{\frac{15}{64}}{z+\frac{1}{2}}$$



Realization of FIR Filters

For FIR filters, a_0 is normalized to unity, and the remaining coefficients $a_k = 0$ for all $k \neq 0$.

$$y[n+K] + a_1 y[n+(K-1)] + \cdots + a_{K-1} y[n+1] + a_K y[n] =$$
$$b_0 x[n+K] + b_1 x[n+(K-1)] + \cdots + b_{K-1} x[n+1] + b_K x[n]$$

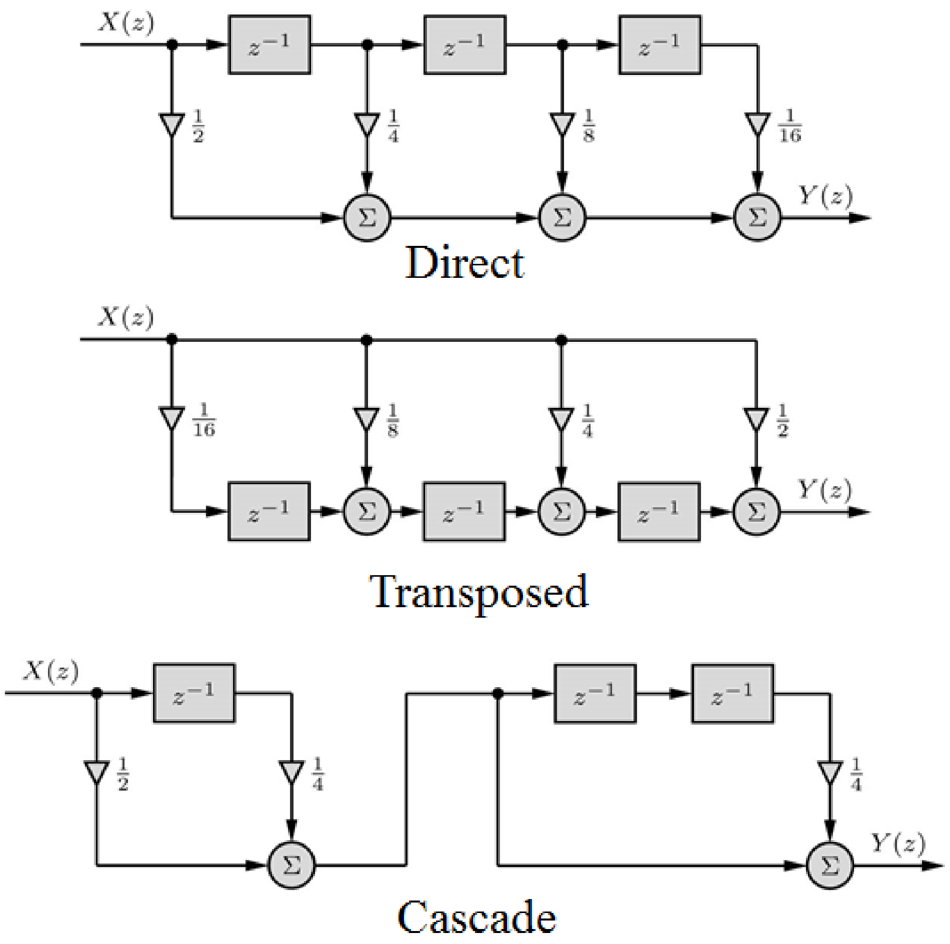
Example: Using direct, transposed direct, and cascade forms, realize the FIR filter

$$y[n] = \frac{1}{2}x[n] + \frac{1}{4}x[n-1] + \frac{1}{8}x[n-2] + \frac{1}{16}x[n-3]$$

For direct and transposed direct $\quad H(z) = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{8}z^{-2} + \frac{1}{16}z^{-3}$

For cascade $\quad H(z) = \dfrac{\frac{1}{2}z^3 + \frac{1}{4}z^2 + \frac{1}{8}z + \frac{1}{16}}{z^3}$

$$H(z) = \left(\frac{\frac{1}{2}z + \frac{1}{4}}{z}\right)\left(\frac{z^2 + \frac{1}{4}}{z^2}\right)$$



Do All Realization Lead to the Same Performance?

Theoretically all realization are equivalent if parameters are implemented with infinite precision.

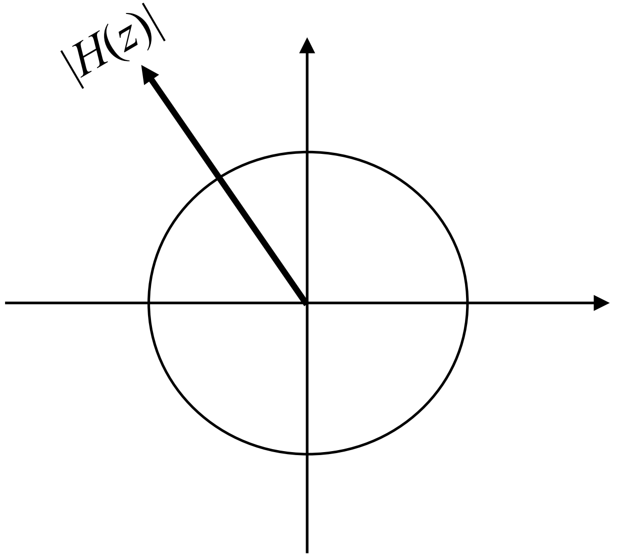
- The finite word-length errors that plague these implementations include coefficient quantization, overflow errors, and round-off errors.
- From a practical viewpoint, parallel and cascade forms using low-order filters minimize finite word-length effects.
- In practice, high-order filters are most commonly realized using a cascade of multiple second-order sections, which are not only easier to design but are also less susceptible to coefficient quantization and round-off errors.

7.6
Frequency Response of Discrete-Time Systems

System Response to Everlasting Sinusoid

The response of a real and asymptotically or BIBO stable LTID system to a sinusoidal input (or exponential) is the same sinusoid (exponential), modified only in gain and phase.

$$x[n] = z_o^n = \left(\gamma e^{j\Omega_o}\right)^n$$



$$x[n] = z_o^n \longrightarrow \boxed{H(z)} \longrightarrow y[n] = H(z_o)z_o^n$$

$$= e^{j\Omega_o n} \qquad\qquad y[n] = H\left(e^{j\Omega_o}\right)e^{j\Omega_o n}$$

$$= \cos(\Omega_o n + \theta) \qquad\qquad y[n] = H\left(e^{j\Omega_o}\right)\cos(\Omega_o n + \theta)$$

$$y[n] = \left|H\left(e^{j\Omega_o}\right)\right|\cos\left(\Omega_o n + \theta + \angle H\left(e^{j\Omega_o}\right)\right)$$

Steady-State Response to Causal Sinusoidal Inputs

$$y_{ss}[n] = \left|H\left(e^{j\Omega}\right)\right|\cos\left[\Omega n + \theta + \angle H\left(e^{j\Omega}\right)\right]u[n]$$

Example

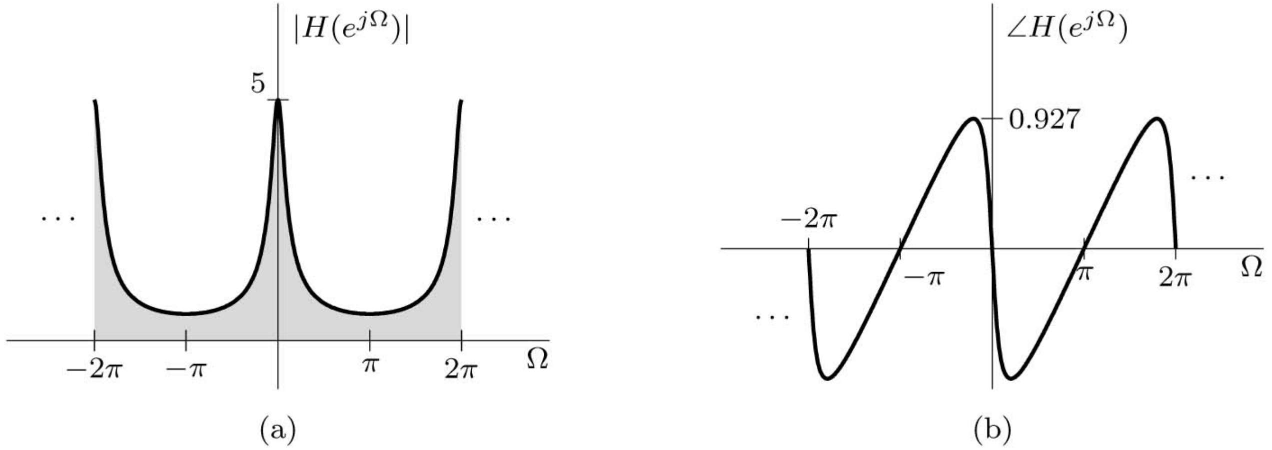
Determine the frequency response $H(e^{j\Omega})$ of the system specified by the equation

$$y[n+1] - 0.8y[n] = x[n+1].$$

Determine the system responses to the inputs (**a**) $x_a[n] = 1^n = 1$, (**b**) $x_b[n] = \cos(\pi/6\, n - 0.2)$, and (**c**) $x_c(t) = \cos(1500t)$ sampled using sampling interval $T = 0.001$.

Solution

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$



$$H(e^{j\Omega}) = \frac{1}{1 - 0.8e^{-j\Omega}} = \frac{1}{1 - 0.8\cos(\Omega) + j0.8\sin(\Omega)}$$

$$|H(e^{j\Omega})| = \frac{1}{\sqrt{1.64 - 1.6\cos(\Omega)}} \quad \text{and} \quad \angle H(e^{j\Omega}) = -\tan^{-1}\left(\frac{0.8\sin(\Omega)}{1 - 0.8\cos(\Omega)}\right)$$

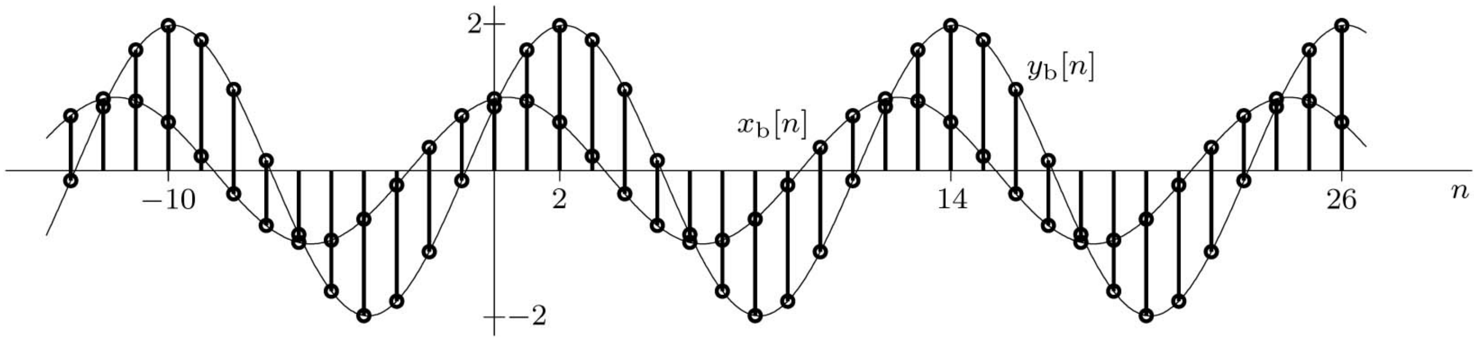
Continue Example

$$|H(e^{j\Omega})| = \frac{1}{\sqrt{1.64 - 1.6\cos(\Omega)}} \quad \text{and} \quad \angle H(e^{j\Omega}) = -\tan^{-1}\left(\frac{0.8\sin(\Omega)}{1 - 0.8\cos(\Omega)}\right)$$

$y_a[n] = 5$

$y_b[n] = 1.983 \cos(\pi/6\, n - 1.116)$

$y_c[n] = 0.8093 \cos(1.5\, n - 0.702)$



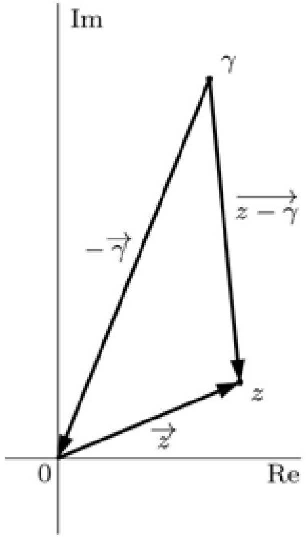

```
Omega = linspace(-2*pi,2*pi,500); H = 1./(1-0.8*exp(-1j*Omega));
subplot(121); plot(Omega,abs(H)); subplot(122); plot(Omega,angle(H));
```

Frequency Response from Pole-Zero Locations

$$H(z) = b_0 \frac{(z-z_1)(z-z_2)\cdots(z-z_K)}{(z-p_1)(z-p_2)\cdots(z-p_K)} = b_0 \frac{\prod_{l=1}^{K}(z-z_l)}{\prod_{k=1}^{K}(z-p_k)}$$

$$H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} = b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_K e^{j\phi_K})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_K e^{j\theta_K})} = b_0 \frac{\prod_{l=1}^{K} r_l e^{j\phi_l}}{\prod_{k=1}^{K} d_k e^{j\theta_k}}$$

$$= b_0 \frac{r_1 r_2 \cdots r_K}{d_1 d_2 \cdots d_K} e^{j[(\phi_1+\phi_2+\cdots+\phi_K)-(\theta_1+\theta_2+\cdots+\theta_K)]}.$$

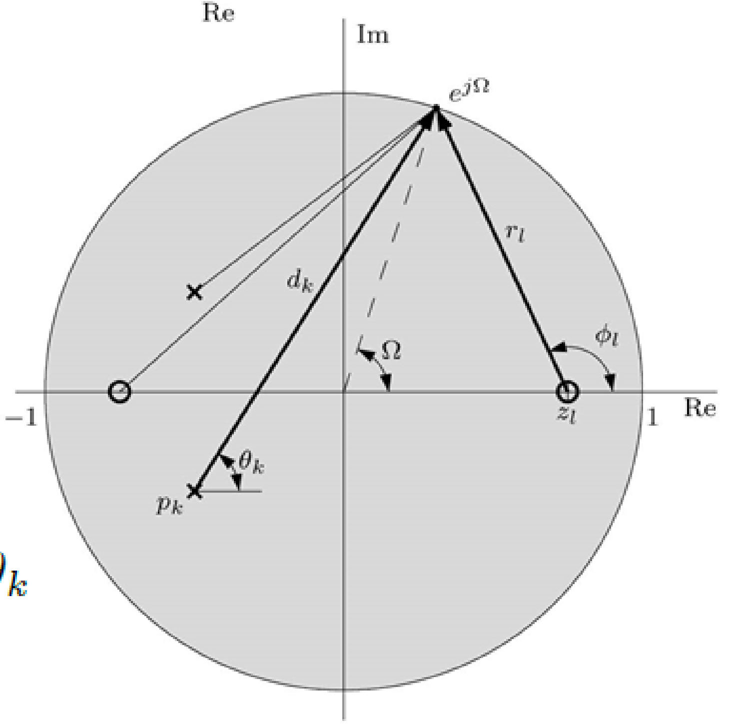


$$|H(e^{j\Omega})| = |b_0| \frac{r_1 r_2 \cdots r_K}{d_1 d_2 \cdots d_K} = |b_0| \frac{\prod_{l=1}^{K} r_l}{\prod_{k=1}^{K} d_k}$$

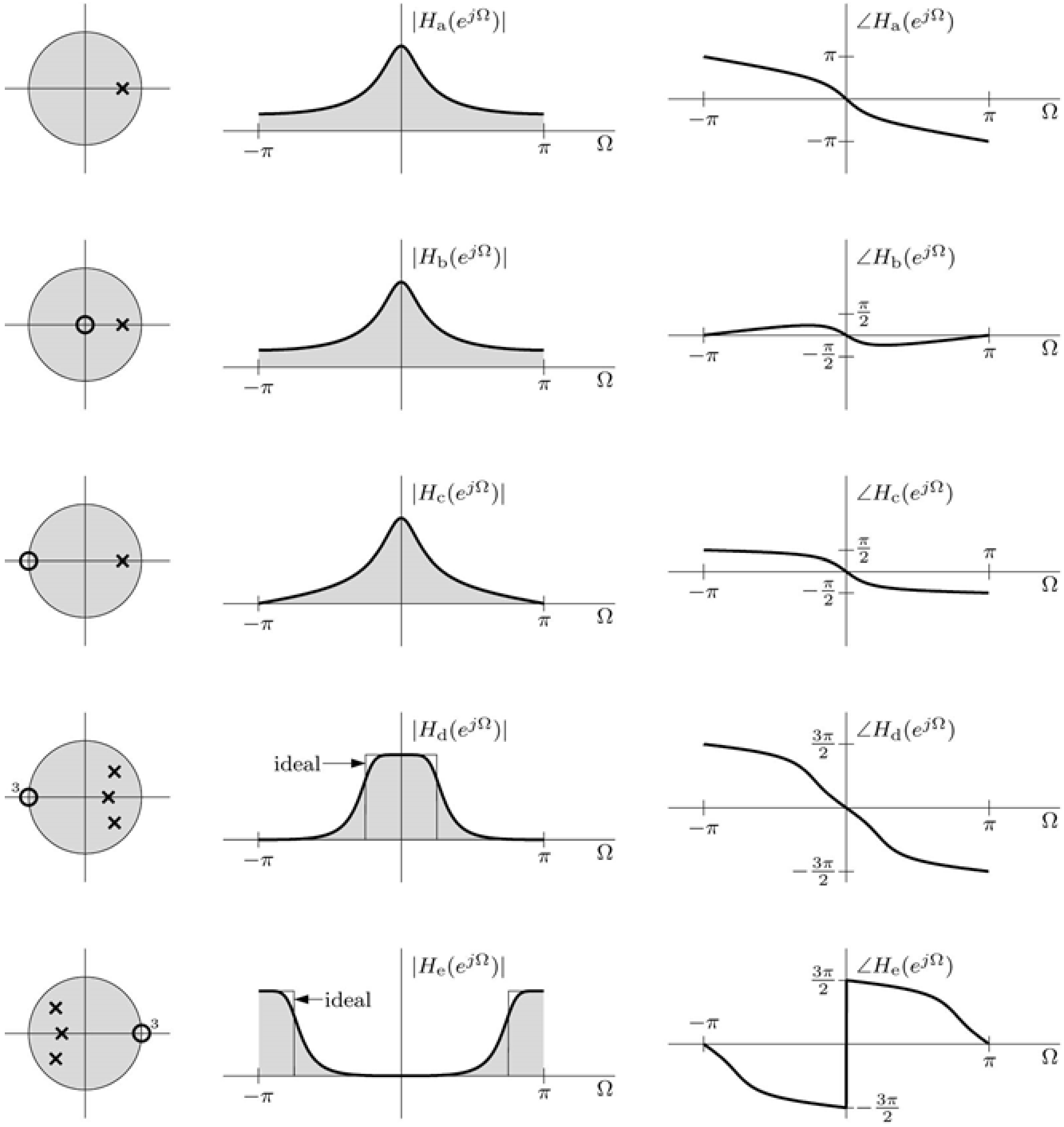
$$= |b_0| \frac{\text{product of the distances of zeros to } e^{j\Omega}}{\text{product of the distances of poles to } e^{j\Omega}},$$

$$\angle H(e^{j\Omega}) = \angle b_0 + (\phi_1 + \phi_2 + \cdots \phi_K) - (\theta_1 + \theta_2 + \cdots + \theta_K) = \angle b_0 + \sum_{l=1}^{K} \phi_l - \sum_{k=1}^{K} \theta_k$$

$$= \angle b_0 + \text{sum of zero angles to } e^{j\Omega} - \text{sum of pole angles to } e^{j\Omega}$$

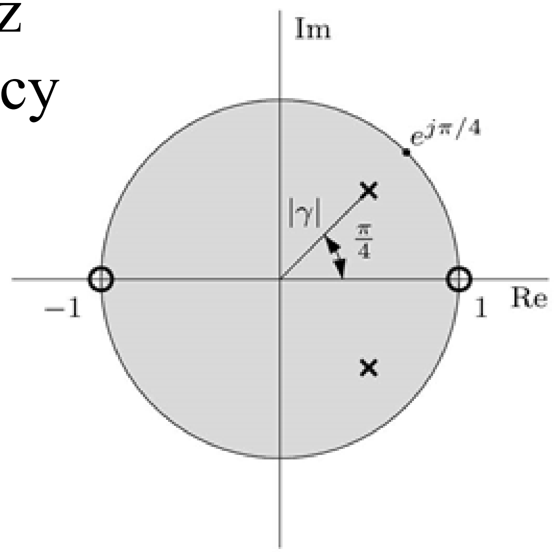


- To enhance the magnitude response at a frequency Ω, place a pole close to the point $e^{j\Omega}$.
- Placing a pole or a zero at the origin does not influence the magnitude response, but it adds angle −Ω (or Ω) to the phase response $\angle H(e^{j\Omega})$.
- To suppress the magnitude response at a frequency Ω, we should place a zero close to the point $e^{j\Omega}$.
- Repeating poles or zeros further enhances their influence.
- Placing a zero close to a pole tends to cancel the effect of that pole on the frequency response (and vice versa).
- For a stable system, all the poles must be located inside the unit circle.



Example

Design a tuned (bandpass) analog filter with zero transmission at 0 Hz and also at the highest frequency f_{max} = 500 Hz. The resonant frequency is required to be 125 Hz.



Solution

$f_s \geq 2f_{max} = 1000$ Hz

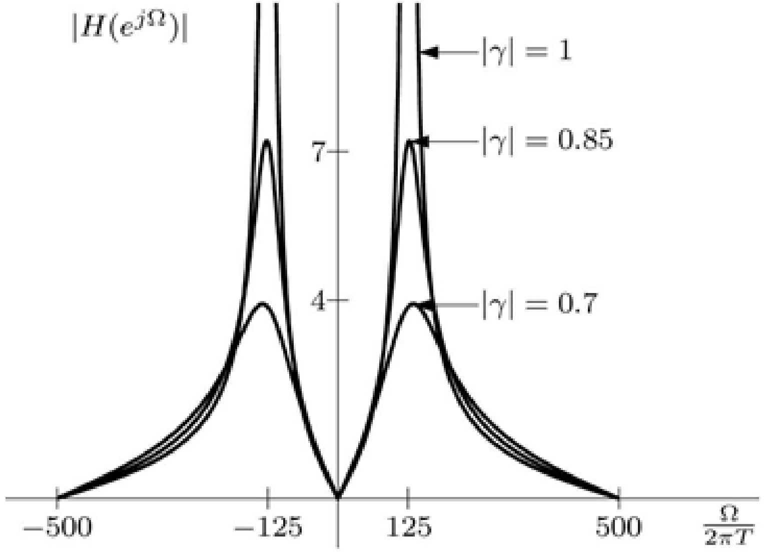
$\Omega = \omega T = 2\pi f / f_s$

f	Ω
0	0
500	π
125	$\pi/4$

$p_1 = |\gamma| e^{j\pi/4}$ $p_2 = |\gamma| e^{-j\pi/4}$

$$H(z) = K \frac{(z-1)(z+1)}{(z - |\gamma|e^{j\pi/4})(z - |\gamma|e^{-j\pi/4})} = K \frac{z^2 - 1}{z^2 - \sqrt{2}|\gamma| z + |\gamma|^2}$$

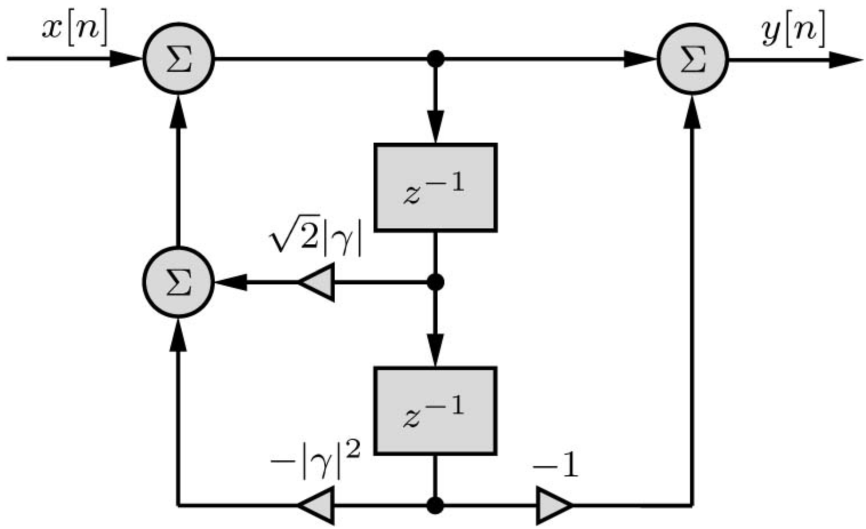
$$H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} = K \frac{e^{j2\Omega} - 1}{e^{j2\Omega} - \sqrt{2}|\gamma| e^{j\Omega} + |\gamma|^2}$$



Continue Example

```
H = @(z,gamma) (z.^2-1)./(z.^2-sqrt(2)*abs(gamma)*z+(abs(gamma))^2);
T = 10^(-3); Omega = linspace(-pi,pi,1001); f = Omega/(2*pi*T); z = exp(j*Omega);
plot(f,abs(H(z,0.7)),f,abs(H(z,0.85)),f,abs(H(z,1))); axis([-500 500 -1 10]);
```

$$H(z) = K\frac{z^2-1}{z^2-\sqrt{2}|\gamma|z+|\gamma|^2}$$



Example

Design a second-order notch filter with zero transmission at 250 Hz and a sharp recovery of gain to unity on both sides of 250 Hz. The highest significant frequency to be processed is $f_{\max} = 400$ Hz.

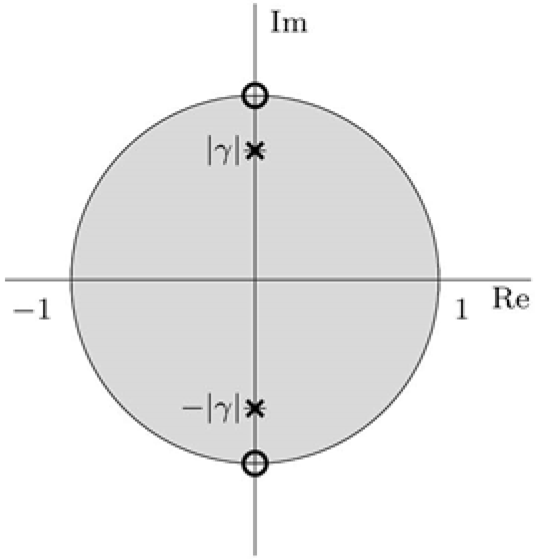
Solution

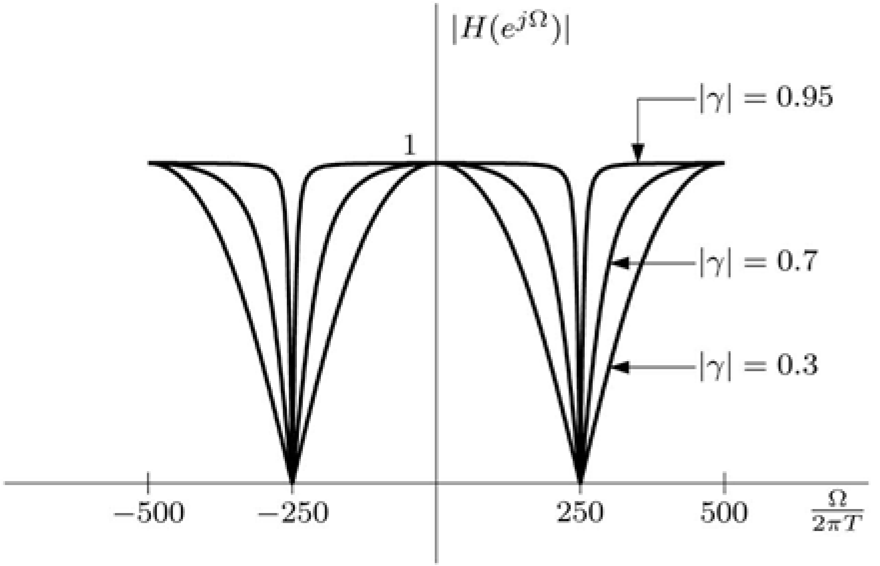
$$F_s \geq 2f_{\max} = 800 \text{ samples/sec}$$

Choose $F_s = 1000$

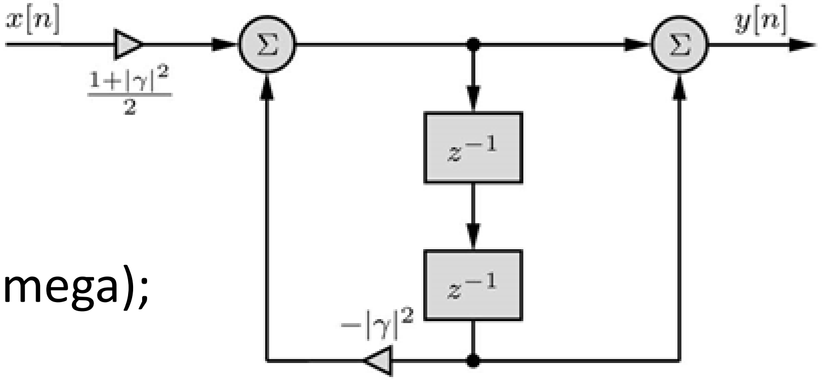
$$\Omega = \omega / F_s = 2\pi f / F_s$$

f	Ω
0	0
400	0.8π
250	$\pi/2$





$$H(z) = K\frac{(z-j)(z+j)}{(z-j|\gamma|)(z+j|\gamma|)} = K\frac{z^2+1}{z^2+|\gamma|^2}$$




```
H = @(z,gamma) (1+(abs(gamma)).^2)/2*(z.^2+1)./(z.^2+(abs(gamma))^2);
T = 10^(-3); Omega = linspace(-pi,pi,1001); f = Omega/(2*pi*T); z = exp(1j*Omega);
plot(f,abs(H(z,0.30)),f,abs(H(z,0.7)),f,abs(H(z,0.95))); axis([-500 500 -1 10]);
```

7.7
Finite Word-Length Effects

Read

Finite Word-Length Effects

$$y[n] = b_o x[n] + b_1 x[n-1] - a_1 y[n-1]$$

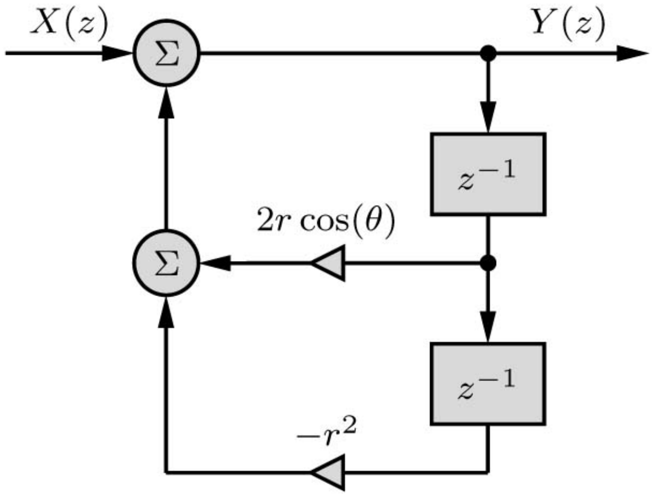
- Word-length is the number of bits used to represent samples of the input $x[n]$, output $y[n]$, and the parameters of the system b_o, b_1, a_1 (filters).
- The shorter the word-length the worst the performance of the system.
- There are many available options that help minimize the adverse effects of finite word lengths.
- Cascade and parallel realizations comprised of low-order sections tend to perform better than single-section direct form realizations.
- Floating-point representations, which are generally less sensitive to finite word-length effects, can be adopted over fixed-point representations.

$$1000{,}000 < b_o < 1000{,}000$$

$$q_{e\text{-}16B} = 15.25$$
$$q_{e\text{-}32B} = 0.0002328$$

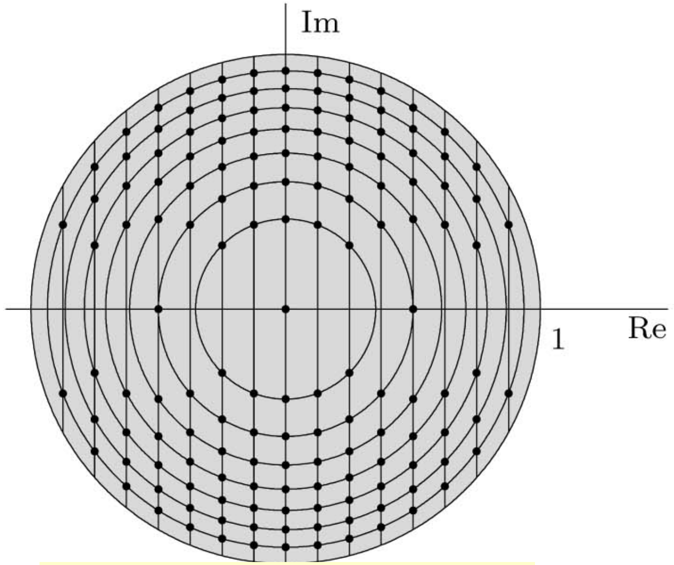
Finite Word-Length Effects on Poles and Zeros

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}}$$

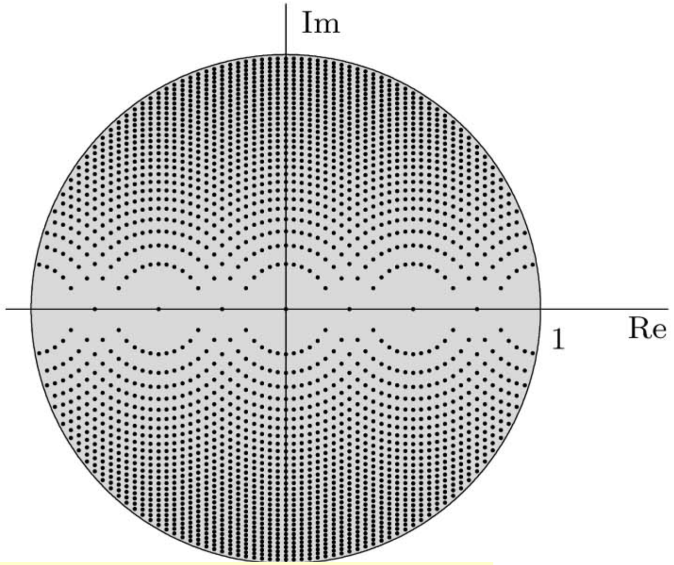


Now, when the system is implemented in hardware, the coefficients $-a_1 = 2r\cos(\theta)$ and $-a_2 = -r^2$ must be quantized.
The range of a_1 and a_2 for stable system are $-2 < a_1 < 2$ and $0 \le a_2 < 1$

There are relatively few pole locations found along the real axis. This means that the direct form realization will have difficulty implementing narrow-band lowpass and highpass filters, which tend to have concentrations of poles near $z = 1$ and $z = -1$, respectively.



4-bit quantization

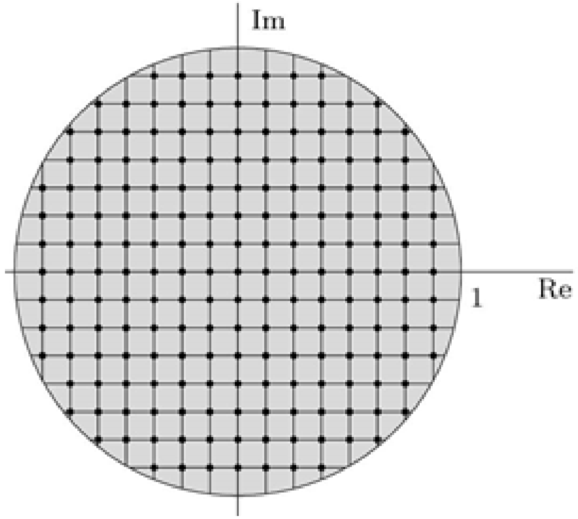


6-bit quantization

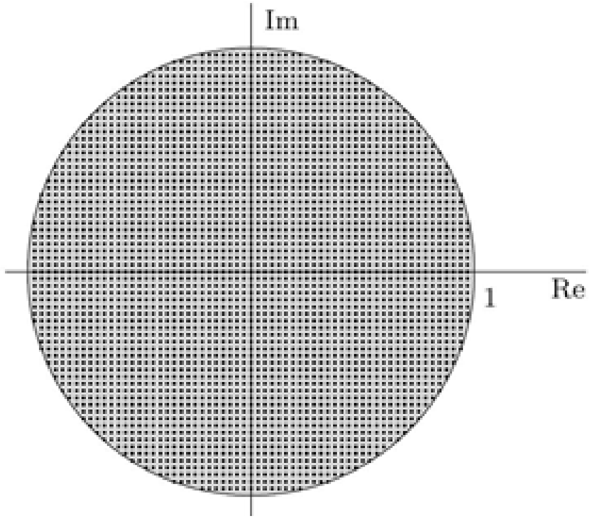
Finite Word-Length Effects on Poles and Zeros

A different design of the same system may reduce the impact of finite word-length without the need to increase the number of bits. The system parameters that need to be quantized (B-bit two's-complement signed number) are $-1 < r\cos(\theta) < 1$ and $-1 < r\sin(\theta) < 1$

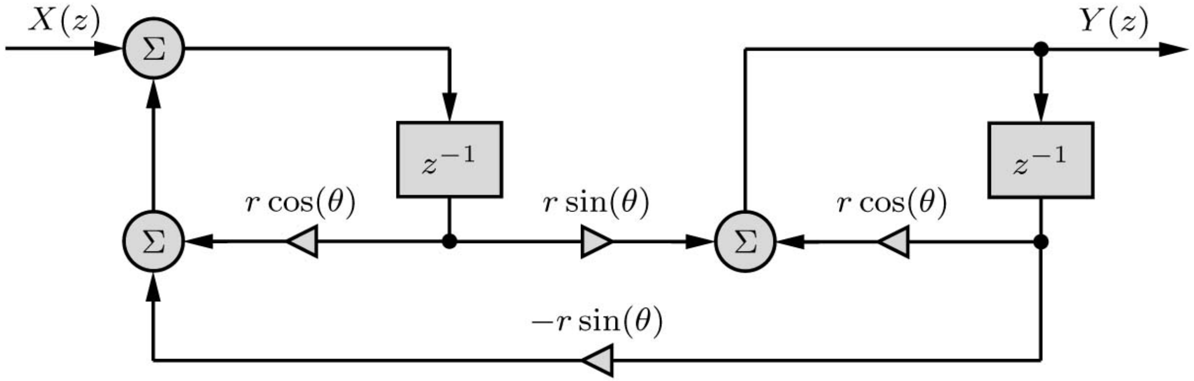
$$H(z) = \frac{1}{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}}$$



4-bit quantization



6-bit quantization



Finite Word-Length Effects on Frequency Response

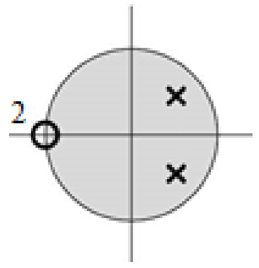
When coefficient quantization alters a system's poles and zeros, the system's frequency response also changes, usually for the worse.

Example: A digital Chebyshev lowpass filter, which has all its zeros at $z = -1$, can be implemented as a cascade of second-order direct form sections, each with transfer function of the form

$$H(z) = \frac{b_0(1 + 2z^{-1} + z^{-2})}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Assuming that the system operates at a sampling frequency of $F_s = 100/\pi$, investigate the effects of 12-, 10-, 8-, and 6-bit coefficient quantization on the magnitude response of the 6th-order Chebyshev lowpass filter described by

Section	b_0	a_1	a_2
1	0.03273793724	-1.81915463768	0.95522952077
2	0.01799913516	-1.80572713311	0.88054034430
3	0.00399530100	-1.82139792759	0.83800435313



Finite Word-Length Effects on Frequency Response

Solution:

The passband frequency ω_p = 12 rad/s and the stopband frequency ω_s = 15 rad/s. Since the passband and stopband frequencies are both well below the folding frequency of 100 rad/s, the filter is relatively narrowband, and all system poles are concentrated near z = 1. As a result, we expect direct form realizations of the system to be somewhat more susceptible than normal to finite word-length effects.

Quantization	Section	b_0	a_1	a_2
12-bit	1	1073×2^{-15}	-1863×2^{-10}	978×2^{-10}
	2	1180×2^{-16}	-1849×2^{-10}	902×2^{-10}
	3	1047×2^{-18}	-1865×2^{-10}	858×2^{-10}
10-bit	1	268×2^{-13}	-466×2^{-8}	245×2^{-8}
	2	295×2^{-14}	-462×2^{-8}	225×2^{-8}
	3	262×2^{-16}	-466×2^{-8}	215×2^{-8}
8-bit	1	67×2^{-11}	-116×2^{-6}	61×2^{-6}
	2	74×2^{-12}	-116×2^{-6}	56×2^{-6}
	3	65×2^{-14}	-117×2^{-6}	54×2^{-6}
6-bit	1	17×2^{-9}	-29×2^{-4}	15×2^{-4}
	2	18×2^{-10}	-29×2^{-4}	14×2^{-4}
	3	16×2^{-12}	-29×2^{-4}	13×2^{-4}

Designating the number of quantization bits as B, each coefficient needs to be represented by an integer in the range -2^{B-1} to $2^{B-1}-1$.

$$b_0 = 0.03273793724$$

12-bit $1073\times2^{-15} = 0.032745361$

8-bit $67\times2^{-11} = 0.032714844$

6-bit $17\times2^{-9} = 0.033203125$

Finite Word-Length Effects on Frequency Response

With 12-bit word-length, coefficient quantization results in a magnitude response that matches almost the ideal magnitude response.

For 6-bit word-length the quantization of the third stage produces two poles at 0.8125 and at 1.0. These two poles will make the system unstable for dc input.