Outline

- Zero-input Response
- Impulse Response $h(t)$
- Convolution
- Zero-State Response
- System Stability
System Response

Initial Conditions are not zeros (stored energy)

\[ x(t) = 0 \quad y(t) = \text{zero input response} \]

Initial Conditions are zeros (No stored energy)

\[ x(t) \quad y(t) = \text{zero state response} \]

Total response = Zero input response + Zero state response
System Response

\[
\frac{d^2 y}{dt^2} + \frac{R}{L} \frac{dy}{dt} + \frac{1}{LC} y = \frac{1}{L} \frac{dx}{dt}
\]

\[
\left( D^2 + \frac{R}{L} D + \frac{1}{LC} \right) y = \frac{1}{L} Dx
\]

For N order system

\[
(D^N + a_1 D^{N-1} + a_2 D^{N-2} + \cdots + a_N) y = \left( b_0 D^M + b_1 D^{M-1} + b_2 D^{M-2} + \cdots + b_M \right) x
\]

What is the zero-input response?
Zero-Input Response

\[(D^N + a_1 D^{N-1} + a_2 D^{N-2} + \cdots + a_N) y(t) = 0\]

A solution to the above differential equation is \(y(t) = ce^{\lambda t}\)

\[Dy = \lambda ce^{\lambda t}\]
\[D^2y = \lambda^2 ce^{\lambda t}\]
\[D^Ny = \lambda^N ce^{\lambda t}\]

\[(\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \cdots + a_N) ce^{\lambda t} = 0\]

Characteristic Polynomial \(Q(\lambda)\)

Characteristic Equation \(Q(\lambda) = 0\)

\[\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \cdots + a_N = 0\]
Next step is to solve the characteristic equation

\[ \lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \cdots + a_N = 0 \]

\[(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0\]

If the characteristic roots \( \lambda \)s are distinct real then a possible solutions to the differential equation are

\[ c_1 e^{\lambda_1 t} \quad c_2 e^{\lambda_2 t} \quad \cdots \quad c_N e^{\lambda_N t} \]

A general solution of the zero-input response is

\[ y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t} \]
The zero-input response is

\[ y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t} \]

**Characteristic Roots:** $\lambda_1 \ldots \lambda_N$

- Also called the natural frequencies or eigenvalues
- For stable system all $\lambda \leq 0$
- Can be real distinct, repeated roots, and/or complex roots

**Characteristic Modes:** $c_1 e^{\lambda_1 t} \ldots c_N e^{\lambda_N t}$, determine the system’s behavior

The constants $c_1, \ldots, c_n$ are arbitrary constants that will be determined by the initial conditions (initial states of the system)
Example

For the LTI system described by the following differential equation \((D^2 + 3D + 2) \cdot y(t) = D \cdot x(t)\)

The initial conditions are \(y(0) = 0\) and \(\dot{y}(0) = -5\)

Find

a) The characteristic equation
b) The characteristic roots
c) The characteristic modes
d) The zero-input response

Answer: \(y(t) = -5e^{-t} + 5e^{-2t}\)
Repeated Characteristic Roots

If the characteristic equation

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_N) = 0$$

has this form

$$(\lambda - \lambda_1)^N = 0,$$

then the $N$ characteristic roots has the same value $\lambda_1$.

The zero-input solution will be

$$y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + c_2 t^2 e^{\lambda_1 t} + \cdots + c_N t^{N-1} e^{\lambda_1 t}$$
Example

For the LTI system described by the following differential equation
\[(D^2 + 6D + 9) y(t) = (3D + 5) x(t)\]
The initial conditions are \(y(0) = 3\) and \(\dot{y}(0) = -7\)

Find
a) The characteristic equation
b) The characteristic roots
c) The characteristic modes
d) The zero-input response

**Answer:** \(y(t) = 3e^{-3t} + 2te^{-3t}\)
Complex Characteristic Roots

If the characteristic equation

\[(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_N) = 0\]

has a complex characteristic root \(\sigma + j\omega\) then its conjugate \(\sigma - j\omega\) is also a characteristic root. This is necessary for the system to be physically realizable.

The zero-input solution for a pair of conjugate roots is

\[y(t) = c_1 e^{(\sigma+j\omega)t} + c_2 e^{(\sigma-j\omega)t}\]

If \(c_1\) and \(c_2\) are complex then \(c_2\) is conjugate of \(c_1\).

\[y(t) = 0.5ce^{j\theta} e^{(\sigma+j\omega)t} + 0.5ce^{-j\theta} e^{(\sigma-j\omega)t}\]

\[y(t) = ce^{\sigma t} \cos(\omega t + \theta)\]
For the LTI system described by the following differential equation \((D^2 + 4D + 40) y(t) = (D + 2) x(t)\)

The initial conditions are \(y(0) = 2\) and \(\dot{y}(0) = 16.78\)

Find

a) The characteristic equation
b) The characteristic roots
c) The characteristic modes
d) The zero-input response

**Answer:** \(y(t) = 4e^{-2t}\cos(6t - \frac{\pi}{3})\)
Meanings of Initial Conditions

- If the output of a system $y = 5x + 3$ then $\frac{dy}{dx} = 5$.

- To find $y$ from the differential equation $\frac{dy}{dx}$, we integrate both side but the answer will be $y = 5x + C$.

- The constant $C$ is the value of $y$ when $x$ was zero. To find $C$ we need an initial condition to tell us what is the value of $y$ when $x$ was zero.

- For $N^{th}$ order differential equation we need $N$ initial conditions (auxiliary conditions) to solve the equation.
The Meaning of $t = 0^-$ and $t = 0^+$

$y(0^-)$: At $t = 0^-$ the input is not applied to the system yet so the output is due only to the initial conditions.

$y(0^+)$: At $t = 0^+$ the input is applied to the system so the output is due the input and the initial conditions.
Impulse Response $h(t)$

$x(t) = \delta(t)$

System

$y(t) = h(t)$
Unit Impulse Response $h(t)$

Reveal system behavior

Depends on the system internal characteristic modes

Helps in finding system response to any input $x(t)$
Derivation of Impulse Response $h(t)$

1- Find the differential equation of the system

$$(D^N + a_1D^{N-1} + \cdots + a_N)y = (b_0D^M + b_1D^{M-1} + \cdots + b_M)x$$

$$Q(D)y(t) = P(D)x(t)$$

2- Find the natural response $y_n(t)$ using the same steps used to find the zero-input response.

$$y_n(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \cdots + c_Ne^{\lambda_N t}$$

3- To find the constants $c$, set all initial conditions to zeros except $N-1$ derivative, set it to equal 1.

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) = \cdots = y_{n}^{N-2}(0) = 0 \quad y_{n}^{N-1}(0) = 1$$

4- $h(t) = [P(D)y_n(t)]u(t)$

If $M = N$ then $h(t) = b_0 \delta(t) + [P(D)y_n(t)] u(t)$
Example

Determine the impulse response $h(t)$ for the system

$$(D^2 + 3D + 2) y(t) = (D + 2) x(t)$$
Zero-State Response

\[ y(t) = x(t) \ast h(t) \]

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \]
Zero-State Response

Any signal can be represented as a train of pulses of different amplitudes and at different locations.

\[
x(t) = \lim_{\Delta \tau \to 0} \sum_{n} x(n\Delta \tau)p(t - n\Delta \tau)
\]

\[
x(t) = \lim_{\Delta \tau \to 0} \sum_{n} x(n\Delta \tau) \frac{p(t - n\Delta \tau)}{\Delta \tau} \Delta \tau
\]

\[
x(t) = \lim_{\Delta \tau \to 0} \sum_{n} x(n\Delta \tau) \delta(t - n\Delta \tau) \Delta \tau
\]
Zero-State Response

input $\Rightarrow$ output

$\delta(t) \Rightarrow h(t)$

$\delta(t - n\Delta\tau) \Rightarrow h(t - n\Delta\tau)$

$[x(n\Delta\tau)\Delta\tau]\delta(t - n\Delta\tau) \Rightarrow [x(n\Delta\tau)\Delta\tau]h(t - n\Delta\tau)$
Zero-State Response

\[
x(t) = \lim_{\Delta \tau \to 0} \sum_{n} x(n\Delta \tau) \delta(t - n\Delta \tau) \Delta \tau
gives \quad y(t) = \sum_{n} x(n\Delta \tau) h(t - n\Delta \tau) \Delta \tau
\]

Each curve represents the output for one value of \( n \).

\[
y(t) = \lim_{\Delta \tau \to 0} \sum_{n} x(n\Delta \tau) h(t - n\Delta \tau) \Delta \tau = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau
\]
Zero-State Response

The Convolution Integral

\[ y(t) = x(t) * h(t) \]
\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \]
\[ y(t) = h(t) * x(t) \]
\[ y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \]
Example

For a LTI system with impulse response $h(t) = e^{-2t} u(t)$, determine the response $y(t)$ for the input $x(t) = e^{-t} u(t)$.

Answer: $y(t) = (e^{-t} - e^{-2t})u(t)$
The Convolution Properties

Commutative: \( x_1 \ast x_2 = x_2 \ast x_1 \)

Associative: \( x_1 \ast [x_2 \ast x_3] = [x_1 \ast x_2] \ast x_3 \)

Distributive: \( x_1 \ast [x_2 + x_3] = [x_1 \ast x_2] + [x_1 \ast x_3] \)

Impulse Convolution: \( x(t) \ast \delta(t) = \int x(\tau)\delta(t - \tau)d\tau = x(t) \)
The Convolution Properties

Shift Property:

if \( x_1(t) \times x_2(t) = c(t) \)
then \( x_1(t - T) \times x_2(t) = x_1(t) \times x_2(t - T) = c(t - T) \)
also \( x_1(t - T_1) \times x_2(t - T_2) = c(t - T_1 - T_2) \)

Width Property:

If the width of \( x_1(t) \) is \( T_1 \) and the width of \( x_2(t) \) is \( T_2 \)
then the width of \( x_1(t) \times x_2(t) \) is \( T_1 + T_2 \)

[Diagram showing convolution of two functions]
Example

Find the total response for the system $D^2y + 3Dy + 2y = Dx$ for input $x(t) = 10e^{-3t}u(t)$ with initial condition $y(0) = 0$ and $\dot{y}(0) = -5$

Answer

$$y(t) = \left(-5e^{-t} + 5e^{-2t}\right) + \left(-5e^{-t} + 20e^{-2t} - 15e^{-3t}\right)$$  
For $t \geq 0$

- **Zero-input Response**
- **Zero-state Response**

$$y(t) = \left(-10e^{-t} + 25e^{-2t}\right) + \left(-15e^{-3t}\right)$$

- **Natural Response**
- **Forced Response**
System Stability

- **External Stability (BIBO)**
  - If the input is bounded then the output is bounded
  \[
  \int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty
  \]

- **Internal Stability (Asymptotic)**
  - If and only if all the characteristic roots are in the LHP
  - Unstable if, and only if, one or both of the following conditions exist:
    - At least one root is in the RHP
    - There are repeated roots on the imaginary axis
  - Marginally stable if, and only if, there are no roots in the RHP, and there are some unrepeated roots on the imaginary axis.
Example

Investigate the asymptotic & BIBO stability of the following systems:

a) \((D+1)(D^2 + 4D + 8) \ y(t) = (D - 3) \ x(t)\)

b) \((D-1)(D^2 + 4D + 8) \ y(t) = (D + 2) \ x(t)\)

c) \((D+2)(D^2 + 4) \ y(t) = (D^2 + D + 1) \ x(t)\)
Example

Find the zero-state response $y(t)$ of the system described by the impulse response $h(t)$ for an input $x(t)$, shown below.
Convolution Demonstration

\[ x(t - \tau) \]

\[ h(\tau) \]